RANK TESTS FOR THE ONE- AND TWO-SAMPLE BIVARIATE LOCATION PROBLEMS

BY

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RANK TESTS FOR THE ONE- AND TWO-SAMPLE
BIVARIATE LOCATION PROBLEMS

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Affine invariant signed-rank tests are proposed for the one- and two-sample bivariate location problems. The test statistics proposed are modifications of the one- and two-sample tests of Blumen and Mardia, respectively. The asymptotic null distributions of the proposed statistics are obtained for the class of elliptically symmetric distributions. In addition, the asymptotic distributions under certain contiguous alternatives are obtained for the proposed statistics, as well as for Blumen's sign statistic and a slightly modified version of Mardia's sign statistic. Comparisons are made between the proposed statistics and several competitors via Pitman asymptotic relative efficiencies and Monte Carlo results. The signed-rank tests proposed appear to be robust. They perform better than the leading competitors when the underlying distribution is light-tailed. For heavy-tailed distributions, Blumen's test and Mardia's

test perform better than the proposed one- and two-sample statistics, respectively. However, the proposed statistics clearly dominate the one- and two-sample Hotelling's T^2 in both light-tailed and heavy-tailed settings and perform virtually as well as Hotelling's T^2 when the underlying population(s) is(are) bivariate normal.

CHAPTER 1
INTRODUCTION

In this study we investigate test statistics for the one- and two-sample bivariate location problems. The rank procedures we propose are modifications of existing procedures used with one and two samples. For the one-sample problem, assume $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independently and identically distributed as $\underset{\sim}{Y}$, where $\underset{\sim}{Y}$ is from a continuous bivariate population. Let $\underset{\sim}{Y}_i = (Y_{1i},Y_{2i})'$, $i = 1,\ldots,n$ and $\underset{\sim}{\theta} = (\theta_1,\theta_2)' = (\text{median } (Y_{1i}), \text{median } (Y_{2i}))'$. We are concerned with the test of

$$H_o: \underset{\sim}{\theta} = \underset{\sim}{\theta}_o \text{ against } H_a: \underset{\sim}{\theta} \neq \underset{\sim}{\theta}_o. \tag{1.1.1}$$

Letting $\underset{\sim}{Z}_i = (Z_{1i}, Z_{2i})' = (\underset{\sim}{Y}_i - \underset{\sim}{\theta}_o)$, $i = 1,\ldots,n$, we see that we can, without loss in generality, consider the test of

$$H_o: \underset{\sim}{\theta} = \underset{\sim}{0} \text{ against } H_a: \underset{\sim}{\theta} \neq \underset{\sim}{0}. \tag{1.1.2}$$

Probably the most well-known procedure to test the hypotheses of 1.1.2 is that proposed by Hotelling. Hotelling's T^2 is defined as

$$T^2 = T^2(\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n) = n\, \bar{\underset{\sim}{Y}}'\hat{\Sigma}^{-1}\bar{\underset{\sim}{Y}},$$

where (1.1.3)

$$\bar{\underset{\sim}{Y}} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{Y}_i, \text{ and } \hat{\Sigma} = \frac{1}{n-1}\sum_{i=1}^{n} (\underset{\sim}{Y}_i-\bar{\underset{\sim}{Y}})(\underset{\sim}{Y}_i-\bar{\underset{\sim}{Y}})'.$$

If Σ, the variance-covariance matrix of $\underset{\sim}{Y}$, is positive definite, then under H_o T^2 is asymptotically Chi-square with two degrees of freedom. If $\underset{\sim}{Y}$ is a bivariate normal random vector, then the null distribution of T^2 is a multiple of the F distribution with two numerator degrees of freedom and n-2 denominator degrees of freedom. This procedure is the likelihood ratio test of 1.1.2 when the underlying distribution is bivariate normal. Desirably, T^2 is invariant with respect to nonsingular linear transformations. That is, if L is a 2×2 nonsingular matrix, then

$$T^2(\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n) = T^2(L\underset{\sim}{Y}_1,\ldots,L\underset{\sim}{Y}_n).$$

For the one-sample problem we shall call this invariance property affine-invariance. (In the two-sample setting affine-invariance will refer to statistics which are invariant with respect to translations of the observations in addition to the nonsingular linear transformations discussed here.) This appealing invariance property ensures that rotations and reflections of the observations about the origin as well as scale changes will not affect the test statistic. Many of these transformations of the observations are equivalent to changes in the covariance structure of the population sampled.

An early competitor of Hotelling's T^2 is a bivariate sign-test due to Hodges (1955). Hodges counts the number of observed vectors falling on each side of a given line through the origin. His statistic, H, is the maximum of the counts obtained, maximizing over all lines through the origin. H is distribution free under

H_o for the class of continuous distributions such that $\underset{\sim}{\theta}$ satisfies

$$P(aY_1+bY_2 > a\theta_1 + b\theta_2) = 1/2, \qquad (1.1.4)$$

for all $(a,b) \neq (0,0)$. Additionally, H is invariant with respect to nonsingular linear transformations. The null distribution of H was tabulated by Hodges for small samples. Later, Joffe and Klotz (1962) simplified Hodges' representation of the null distribution yielding an expression for the asymptotic null distribution as well.

Another affine-invariant statistic was proposed by Blumen (1958). His sign test statistic is obtained by conditioning on the vectors $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ and their respective projections through the origin, namely $-\underset{\sim}{Y}_1,\ldots,-\underset{\sim}{Y}_n$. The angles between adjacent vectors are adjusted so that each of these standardized angles is π/n radians. The distances of the observed points from the origin are also standardized so that each vector has unit length. This standardization process is illustrated in Figures 1.1 through 1.3. The three vectors $\underset{\sim}{Y}_1$, $\underset{\sim}{Y}_2$, and $\underset{\sim}{Y}_3$ shown in Figure 1.1 are displayed again in Figure 1.2 along with their projections through the origin, $-\underset{\sim}{Y}_1$, $-\underset{\sim}{Y}_2$, and $-\underset{\sim}{Y}_3$. In Figure 1.3 the angles between adjacent vectors are standardized as are the lengths of each vector. The standardized vectors are denoted by $\underset{\sim}{Y}_1^s$, $\underset{\sim}{Y}_2^s$, and $\underset{\sim}{Y}_3^s$. This standardization method is used to preserve affine-invariance. (This fact will be established clearly in Section 2 of Chapter 2.) To obtain Blumen's statistic, the center of gravity of the standardized versions of the vectors $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ is found. His test statistic is proportional to the squared distance from the origin to the center of gravity. The further the center of gravity is from

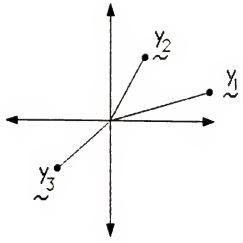

Figure 1.1 Observed Vectors

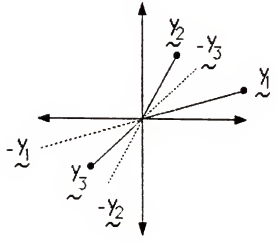

Figure 1.2 Observed Vectors and Their Projections through the Origin

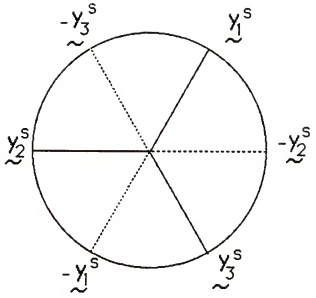

Figure 1.3 Standardized Vectors

the origin, the greater the evidence that H_a is true. Blumen's statistic is distribution free under H_o for the class of distributions given by 1.1.4. Blumen shows that the asymptotic null distribution of his statistic is Chi-square with two degrees of freedom. The null distribution of Blumen's statistic was tabulated by Klotz (1964) for small samples. Our proposed signed-rank, affine-invariant statistic is a modification of Blumen's sign statistic. We will present a more formal discussion of Blumen's statistic in the following chapter as an aid to the introduction of our modification. Subsequently, additional properties of Blumen's test will be derived.

Several one-sample statistics have been proposed which are not affine-invariant. Bennett proposes two procedures. The first method (Bennett, 1962) considers the signs of the components of (Z_{1i}, Z_{2i}), $i = 1,\ldots,n$. Letting

$$S_{11} = \sum_{i=1}^{n} I[Z_{1i} > 0,\ Z_{2i} > 0],$$

$$S_{10} = \sum_{i=1}^{n} I[Z_{1i} > 0,\ Z_{2i} < 0],$$

$$S_{01} = \sum_{i=1}^{n} I[Z_{1i} < 0,\ Z_{2i} > 0],$$

and

$$S_{00} = \sum_{i=1}^{n} I[Z_{1i} < 0,\ Z_{2i} < 0],$$

Bennett's first statistic is given by

$$B1 = \frac{(S_{11}-S_{00})^2}{(S_{11}+S_{00})} + \frac{(S_{10}-S_{01})^2}{(S_{10}+S_{01})} . \qquad (1.1.5)$$

The asymptotic null distribution of B1 is Chi-square with two degrees of freedom and hence B1 can be used for a large sample test. Bennett's procedure was introduced as a general p-variate procedure, p≥2. His suggested bivariate procedure, given here, is identical to the large sample test procedure proposed by Chatterjee (1966). Chatterjee also develops a small sample conditionally distribution-free procedure based on S_{11}, S_{10}, S_{01}, and S_{00}. To do this, Chatterjee uses the result that conditional on $S_{11}+S_{00}$ and $S_{10}+S_{01}$, S_{11} and S_{01} are independent and distributed as binomial random variables with parameters $(S_{11}+S_{00}, 1/2)$ and $(S_{10}+S_{01}, 1/2)$, respectively. The second procedure proposed by Bennett (1964) is a signed-rank test generalizing Wilcoxon's univariate signed-rank test. Bennett bases his test on the univariate signed-ranks of $\{Z_{11},\ldots,Z_{1n}\}$ and $\{Z_{21},\ldots,Z_{2n}\}$. Let

$$U^+ = \sum_{i=1}^{n} \text{Rank}(|Z_{1i}|)I[Z_{1i} > 0] - \tfrac{1}{4} n(n+1)$$

and (1.1.6)

$$V^+ = \sum_{i=1}^{n} \text{Rank}(|Z_{2i}|)I[Z_{2i} > 0] - \tfrac{1}{4} n(n+1),$$

where $\text{Rank}(|Z_{ij}|)$ denotes the rank of $|Z_{ij}|$ among $|Z_{i1}|,\ldots,|Z_{in}|$, $i = 1, 2$. Bennett's signed-rank statistic is a quadratic form based on U^+ and V^+. His statistic is asymptotically Chi-square, under H_o, with two degrees of freedom. In addition to the quadratic forms proposed by Bennett and Chatterjee, Bickel (1965) and Sen and Puri (1967) also consider quadratic functions of both the coordinate-wise sign statistics and the coordinate-wise signed-rank statistics. The class of statistics examined by Sen and Puri is the most general, allowing functions of the coordinate-wise ranks to be used in place of the ranks themselves. Both Bickel (1965) and Sen and Puri (1967) present asymptotic properties of the statistics considered. None of the quadratic forms discussed are affine-invariant. Hence, the performances of these procedures vary with changes in the underlying covariance structure of the population and with changes in the direction of the shift of $\underset{\sim}{\theta}$ from the origin.

Another generalization of the univariate Wilcoxon signed-rank statistic is given by Killeen and Hettmansperger (1972). They consider the perpendicular projection of $\underset{\sim}{Z}_i$, $i = 1,\ldots,n$, onto the line, $l(t)$, which is a counterclockwise rotation of t radians of the horizontal axis. Letting $Z_1(t),\ldots,Z_n(t)$ be the projected points, Killeen and Hettmansperger then calculate the Wilcoxon signed-rank statistic, $W(t)$, based on the relative positions of $Z_1(t),\ldots,Z_n(t)$ on the line, $l(t)$, whose center is at the origin. The test statistic suggested is $\max_{0\le t\le 2\pi} W(t)$. Although this statistic has high Bahadur efficiency, neither its small sample nor its asymptotic null distributions have been found, and hence its utility is questionable.

Brown and Hettmansperger (1985) propose a statistic based on the gradient of Oja's measure of scatter (Oja, 1983). Letting $A(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j, \underset{\sim}{\theta})$ denote the area of the triangle formed by $\underset{\sim}{Y}_i$, $\underset{\sim}{Y}_j$, and $\underset{\sim}{\theta}$ as vertices, define

$$T(\underset{\sim}{\theta}) = \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} A(\underset{\sim}{Y}_i, \underset{\sim}{Y}_j, \underset{\sim}{\theta}). \tag{1.1.7}$$

The value $\hat{\underset{\sim}{\theta}}$ that minimizes $T(\underset{\sim}{\theta})$ is Oja's generalized median for bivariate samples. Brown and Hettmansperger propose the use of

$$Q_n(\underset{\sim}{\theta}) = \nabla T(\underset{\sim}{\theta}), \tag{1.1.8}$$

the vector of partial derivatives of $T(\underset{\sim}{\theta})$. The generalized median of Oja is the value $\hat{\underset{\sim}{\theta}}$ such that $|Q_n(\hat{\underset{\sim}{\theta}})|$ is minimized. The proposed large sample test statistic for the hypotheses of 1.1.2 is a quadratic form of

$$\underset{\sim}{S} = \sum_{i=1}^{n} Q_{2n}(-\underset{\sim}{Y}_i), \tag{1.1.9}$$

where $Q_{2n}(-\underset{\sim}{Y}_i)$ is calculated using the observations $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$, and their projections through the origin, $-\underset{\sim}{Y}_1,\ldots,-\underset{\sim}{Y}_n$.

Also invariant with respect to linear transformations is a statistic proposed by Dietz (1982). She suggests using the sign and signed-rank procedures proposed by Bickel after a transformation is applied to obtain affine-invariance. Each of the tests proposed by Brown and Hettmansperger (1985) and Dietz (1982) are quite difficult

to apply. The asymptotic efficiencies of their procedures have not been determined.

We now consider bivariate tests for location in the two-sample setting. Many of the procedures proposed are natural extensions of the one-sample procedures discussed earlier in this section. Assume $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$ and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independent random samples from continuous bivariate populations with distribution functions $G_1(\underset{\sim}{x})$ and $G_2(\underset{\sim}{y}) = G_1(\underset{\sim}{y}-\underset{\sim}{\theta})$, respectively. We shall consider tests of

$$H_o: \ \underset{\sim}{\theta} = \underset{\sim}{0} \text{ against } H_a: \ \underset{\sim}{\theta} \neq \underset{\sim}{0}. \tag{1.1.10}$$

The normal theory affine-invariant test, extending the one-sample Hotelling's T^2, takes the form

$$T_2^2 = \frac{mn}{N}(\underset{\sim}{\bar{X}}-\underset{\sim}{\bar{Y}})'S^{-1}(\underset{\sim}{\bar{X}}-\underset{\sim}{\bar{Y}}), \tag{1.1.11}$$

where

$$N = m+n, \qquad \underset{\sim}{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m} \underset{\sim}{X}_i, \qquad \underset{\sim}{\bar{Y}} = \frac{1}{n}\sum_{j=1}^{n} \underset{\sim}{Y}_j,$$

and

$$S = \frac{1}{N-2}\{\sum_{i=1}^{m}(\underset{\sim}{X}_i-\underset{\sim}{\bar{X}})(\underset{\sim}{X}_i-\underset{\sim}{\bar{X}})' + \sum_{j=1}^{n}(\underset{\sim}{Y}_j-\underset{\sim}{\bar{Y}})(\underset{\sim}{Y}_j-\underset{\sim}{\bar{Y}})'\}.$$

The asymptotic null distribution of T_2^2 is Chi-square with two degrees of freedom. When both populations are bivariate normal, a small sample test can be obtained using the exact null distribution of

T_2^2, which is a multiple of the F distribution with two numerator degrees and N-3 denominator degrees of freedom.

A generalization of Blumen's one-sample statistic is given by Mardia (1967). Mardia's affine-invariant sign statistic uses the centered sample observations,

$$\underset{\sim}{X}_1-\bar{\underset{\sim}{Z}},\ldots,\underset{\sim}{X}_m-\bar{\underset{\sim}{Z}}, \text{ and } \underset{\sim}{Y}_1-\bar{\underset{\sim}{Z}},\ldots,\underset{\sim}{Y}_n-\bar{\underset{\sim}{Z}},$$

where $\bar{\underset{\sim}{Z}}$ refers to the arithmetic mean of all m+n = N vectors. The standardization process of Blumen's is applied to the N centered vectors. The center of gravity of the standardized versions of the vectors from the first sample is then obtained. Again, the squared distance from the center of gravity to the origin is the basis of the test statistic used. The precise definition of Mardia's statistic is given in Chapter 5. Our proposed two-sample signed-rank statistic will be seen to be a modification of Mardia's statistic. It is a natural extension of our one-sample procedure to two samples.

A distribution-free procedure designed to detect broad alternatives was proposed by Vincze (1961). Vincze suggests projecting each vector onto a randomly selected line through the origin. The Kolmogorov-Smirnov two-sample test statistic is then applied to the projected vectors using their relative positions on the line selected. As a precursor to Killeen and Hettmansperger's one-sample procedure, Vincze also suggests that it may be desirable to maximize the Kolmogorov-Smirnov deviations of the projected observations over all lines through the origin. However, the null distribution when maximizing has not been found.

Chatterjee and Sen (1964) extend the Wilcoxon Rank Sum Test and Mood's Median Test to the bivariate setting. The rank sum test uses the centered Wilcoxon Rank Sums of the marginal variates:

$$S_1 = \sum_{j=1}^{m} \text{Rank}(X_{1j}) - m(\frac{N+1}{2})$$

and (1.1.12)

$$S_2 = \sum_{j=1}^{m} \text{Rank}(X_{2j}) - m(\frac{N+1}{2}),$$

where $\text{Rank}(X_{ij})$ denotes the rank of X_{ij}, among $\{X_{i1},\ldots,X_{im}, Y_{i1},\ldots,Y_{in}\}$, $i = 1, 2$. The large sample test statistic is a quadratic form based on S_1 and S_2. A small sample conditional test based on S_1 and S_2 is also described. This test is also examined by David and Fix (1961) in a discussion of the comparison of two regression lines.

To describe the extension of Mood's median test, let

$$M_1 = \text{median of } \{X_{11},\ldots,X_{1m}, Y_{11},\ldots,Y_{1n}\}$$

and

$$M_2 = \text{median of } \{X_{21},\ldots,X_{2m}, Y_{21},\ldots,Y_{2n}\}.$$

Define

$$M_{11} = \sum_{i=1}^{m} I[X_{1i} \leq M_1, X_{2i} \leq M_2]$$

$$M_{01} = \sum_{i=1}^{m} I[X_{1i} > M_1, X_{2i} \leq M_2]$$

$$M_{10} = \sum_{i=1}^{m} I[X_{1i} \leq M_1, X_{2i} > M_2]$$

$$M_{00} = \sum_{i=1}^{m} I[X_{1i} > M_1, X_{2i} > M_2]$$

and

$$C = \sum_{i=1}^{m} I[X_{1i} \leq M_1, X_{2i} \leq M_2] + \sum_{j=1}^{n} I[Y_{1j} \leq M_1, Y_{2j} \leq M_2].$$

The bivariate median test statistic of Chatterjee and Sen is given by

$$CS_2 = \frac{N^2}{2mn}\left(\frac{(M_{11}-M_{00})^2}{C} + \frac{(M_{10}-M_{01})^2}{[|\frac{N}{2}|]-C}\right)$$

where $[|x|]$ denotes the greatest integer less than or equal to x. A conditional small sample test and a large sample Chi-square test are discussed.

Suguira (1965) and Bhapkar (1966) consider multivariate multisample extensions of the Wilcoxon Rank Sum Statistic. In the bivariate two-sample setting, these large sample tests are functions of the component-wise Wilcoxon Rank Sum Statistics. Puri and Sen (1966) and Tamura (1966) discuss a more general class of Chernoff-Savage type statistics for the multivariate multisample location and scale problem. As in the one-sample setting, score functions of the

coordinate-wise ranks may be included. Both articles discuss asymptotic properties of the given class.

Finally we mention that Brown and Hettmansperger (1985) have also discussed a two-sample test based on Oja's measure of scatter. The large sample test statistic used is a quadratic function of

$$\underset{\sim}{S}_2 = \sum_{i=1}^{m} Q_N(\underset{\sim}{X}_i),$$

where Q_N is the gradient function defined by 1.1.7 and 1.1.8 as applied to the combined sample of N observations.

With the exception of Hotelling's T_2^2, Mardia's test statistic, the test statistic of Brown and Hettmansperger, and the two-sample procedure we shall propose, the other procedures mentioned are not affine-invariant.

The one-sample statistic we propose is defined in Chapter 2. Its invariance with respect to nonsingular linear transformations is also verified. In Chapter 3 we derive the asymptotic null distribution of the one-sample test. The asymptotic distributions under Pitman alternatives of our test and the test of Blumen are then determined. Comparisons of several one-sample procedures are made in Chapter 4 via Pitman asymptotic relative efficiencies and Monte Carlo results. The introduction of our two-sample test and the verification of its invariance with respect to nonsingular linear transformations is given in Chapter 5. Chapter 6 presents the asymptotic distribution of the proposed two-sample test statistic under the null hypothesis and under a sequence of Pitman alternatives. Also given are the corresponding

asymptotic distributions of a "centered" version of Mardia's two-sample test. Finally several two-sample test statistics are compared in Chapter 7 through asymptotic relative efficiencies and some Monte Carlo studies.

CHAPTER 2
A BIVARIATE ONE-SAMPLE TEST FOR LOCATION

2.1 Definition of the Test Statistic

In order to motivate the test statistic we are proposing we will first describe, more formally than in Chapter 1, the test statistic proposed by Blumen. We assume that $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independent and identically distributed as $\underset{\sim}{Y}$, where $\underset{\sim}{Y}$ is from a continuous bivariate distribution with median $\underset{\sim}{\theta}$ satisfying 1.1.4. Let $\phi_1',\ldots,\phi_n'$ denote the angles between the positive half of the horizontal axis and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$, respectively. Here $0 \le \phi_i' < 2\pi$, $i = 1,\ldots,n$. Let

$$\phi_i = \begin{cases} \phi_i', & \text{if } 0 \le \phi_i' \le \pi, \\ \phi_i'-\pi, & \text{if } \pi < \phi_i' < 2\pi, \end{cases}$$

and (2.1.1)

$$a_i = \begin{cases} 1, & \text{if } 0 \le \phi_i' \le \pi, \\ -1, & \text{if } \pi < \phi_i' < 2\pi, \end{cases}$$

for $i = 1,\ldots,n$. For each i, let $\text{Rank}(\phi_i)$ denote the rank of ϕ_i among $\phi_1,\ldots,\phi_n$. (That is, $\text{Rank}(\phi_i)$ equals the number of angles

$\phi_1,\ldots,\phi_n$ that are less than or equal to ϕ_i.) If we denote the center of gravity of the standardized observations, as described in Chapter 1, by (C_B, S_B), then the center of gravity can be defined by the following:

$$C_B = \frac{1}{n}\sum_{i=1}^{n} a_i \cos\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)$$

and (2.1.2)

$$S_B = \frac{1}{n}\sum_{i=1}^{n} a_i \sin\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right).$$

Blumen's sign statistic is given by

$$B = 2n(C_B^2 + S_B^2) \tag{2.1.3}$$

With Blumen's statistic as background, we can now introduce our proposed signed-rank statistic. We begin as Blumen did by considering the observed vectors and their projections through the origin. The angles between adjacent vectors are standardized as in Blumen's statistic, but the lengths of the vectors are not. Instead, we use a measure of the length of each vector that is invariant with respect to linear transformations of the observations. Let

$$D_i = \underset{\sim}{Y}_i' \tilde{\Sigma}^{-1} \underset{\sim}{Y}_i, \qquad \text{for } i = 1,\ldots,n,$$

where (2.1.4)

$$\tilde{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{Y}_i \underset{\sim}{Y}_i'.$$

The matrix $\underset{\sim}{\Sigma}$ is a consistent estimation of $E[\underset{\sim}{Y}\underset{\sim}{Y}']$, the covariance matrix of $\underset{\sim}{Y}$ under H_o, provided $\underset{\sim}{Y}$ is symmetrically distributed about its median. We now weight the i^{th} terms in the averages C_B and S_B of Blumen's statistic by the rank of D_i among $D_1,\ldots,D_n$ and define (C_{MB}, S_{MB}) by

$$C_{MB} = \frac{1}{n}\sum_{i=1}^{n} a_i \cos\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)\text{Rank}(D_i)$$

and (2.1.5)

$$S_{MB} = \frac{1}{n}\sum_{i=1}^{n} a_i \sin\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)\text{Rank}(D_i).$$

Our modification of Blumen's statistic, which we shall denote by MB, is given by

$$MB = 6n^{-1}(C_{MB}^2 + S_{MB}^2). \tag{2.1.6}$$

2.2 Affine-Invariance of MB

In this section we will verify that our proposed statistic is invariant with respect to nonsingular linear transformations of the observations. If L is a 2×2 nonsingular matrix, we will show that $MB(\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n) = MB(L\underset{\sim}{Y}_1,\ldots,L\underset{\sim}{Y}_n)$, where $MB(\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_n)$ denotes the proposed statistic, MB, applied to the observations $\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_n$. This result is established via a sequence of three lemmas:

Lemma 2.2.1. MB is invariant with respect to a rotation of the observations.

Proof. Let $\phi_1,\ldots,\phi_n$ be, as defined in 2.1.1, the angles formed by the vectors projected above the horizontal axis with respect to the positive half of the horizontal axis. For simplicity, let us call $\phi_1,\ldots,\phi_n$ the projected angles associated with $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$, respectively. Now consider a clockwise rotation of the observations such that a single observed vector crosses the horizontal axis. (See Figures 2.1 and 2.2.) Let $\underset{\sim}{Y}_1^*,\ldots,\underset{\sim}{Y}_n^*$ be the vectors corresponding to $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$, respectively, after the rotation. Let $\phi_1^*,\ldots,\phi_n^*$ be the projected angles associated with $\underset{\sim}{Y}_1^*,\ldots,\underset{\sim}{Y}_n^*$, respectively. As can be seen in Figures 2.3 and 2.4, the relationship between the rank of ϕ_i^* and the rank of ϕ_i is given by

$$\text{Rank}(\phi_1^*) = \begin{cases} n, & \text{if } \text{Rank}(\phi_1)=1, \\ \text{Rank}(\phi_i)-1, & \text{otherwise.} \end{cases}$$

Since a rotation of the vectors $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ can be represented as $R\underset{\sim}{Y}_1,\ldots,R\underset{\sim}{Y}_n$, for some positive definite 2×2 matrix R, and since

$$\sum_{i=1}^{n} (R\underset{\sim}{Y}_i)(R\underset{\sim}{Y}_i)' = R\,\tilde{\Sigma}\,R' \qquad (2.2.2)$$

and

$$(R\underset{\sim}{Y}_i)'(R\tilde{\Sigma}R')^{-1}(R\underset{\sim}{Y}_i) = \underset{\sim}{Y}_i'\,\tilde{\Sigma}^{-1}\,\underset{\sim}{Y}_i = D_i,$$

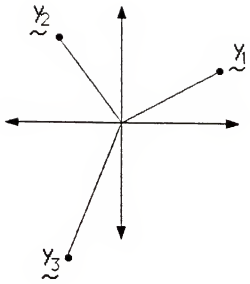

Figure 2.1 Observed Vectors

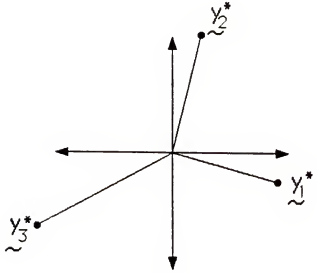

Figure 2.2 Rotated Vectors

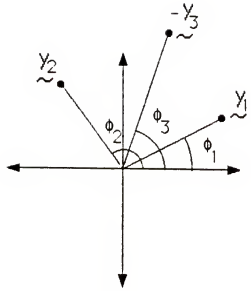

Figure 2.3 Projected Angles of Observed Vectors

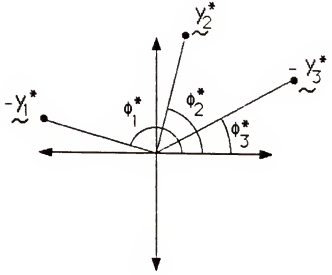

Figure 2.4 Projected Angles of Rotated Vectors

for i = 1,...,n, we see that the standardized distances, D_i, are unaffected by such a rotation. (See also Theorem 2.2.7.) Let $a_1^*,\ldots,a_n^*$ be the sign functions associated with the rotated observations, $\underset{\sim}{Y}_1^*,\ldots,\underset{\sim}{Y}_n^*$, respectively, defined as in 2.1.1. The only sign function which is changed through a rotation is that associated with the vector which is rotated across the horizontal axis. That is,

$$a_i^* = \begin{cases} -a_i, & \text{if } \text{Rank}(\phi_i)=1, \\ a_i, & \text{otherwise.} \end{cases}$$

Given the above notation, our signed-rank statistic when applied to the rotated observations, $Y_1^*,\ldots,Y_n^*$ becomes

$$MB = 6n^{-1}(C^{*2}+S^{*2}),$$

where

$$C^* = \frac{1}{n}\sum_{i=1}^{n} a_i^* \cos\left(\frac{\pi \text{Rank}(\phi_i^*)}{n}\right)\text{Rank}(D_i),$$

and (2.2.3)

$$S^* = \frac{1}{n}\sum_{i=1}^{n} a_i^* \sin\left(\frac{\pi \text{Rank}(\phi_i^*)}{n}\right)\text{Rank}(D_i).$$

If the k^{th} vector is the one which is rotated across the horizontal axis, that is $\text{Rank}(\phi_k) = 1$, then using the relationships described above we can write

$$C^* = \frac{1}{n} \sum_{\substack{i=1 \\ i \neq k}}^{n} a_i \cos\left(\frac{\pi(\text{Rank}(\phi_i)-1)}{n}\right)\text{Rank}(D_i) + \frac{1}{n}\left(-a_k \cos\left(\frac{\pi n}{n}\right)\right)\text{Rank}(D_k).$$

Noting that

$$\cos(\pi) = -\cos(0) = -\cos\left(\frac{\pi(\text{Rank}(\phi_k)-1)}{n}\right),$$

we have

$$C^* = \frac{1}{n} \sum_{i=1}^{n} a_i \cos\left(\frac{\pi(\text{Rank}(\phi_i)-1)}{n}\right)\text{Rank}(D_i)$$

and, likewise,

$$S^* = \frac{1}{n} \sum_{i=1}^{n} a_i \sin\left(\frac{\pi(\text{Rank}(\phi_i)-1)}{n}\right)\text{Rank}(D_i).$$

We will now make use of two elementary trigonometric identities:

$$\cos(b-d) = \cos(b)\cos(d) + \sin(b)\sin(d)$$

and (2.2.4)

$$\sin(b-d) = \sin(b)\cos(d) - \cos(b)\sin(d).$$

Taking $b = \frac{\pi \text{Rank}(\phi_i)}{n}$ and $d = \frac{\pi}{n}$, we can write

$$C^* = \frac{1}{n} \sum_{i=1}^{n} a_i \left\{\cos\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)\sin\left(\frac{\pi}{n}\right)\right\}\text{Rank}(D_i)$$

and

$$S^* = \frac{1}{n}\sum_{i=1}^{n} a_i\{\sin(\frac{\pi \text{Rank}(\phi_i)}{n})\cos(\frac{\pi}{n}) - \cos(\frac{\pi \text{Rank}(\phi_i)}{n})\sin(\frac{\pi}{n})\}\text{Rank}(D_i).$$

That is,

$$C^* = \cos(\tfrac{\pi}{n})C_{MB} + \sin(\tfrac{\pi}{n})S_{MB}$$

and

$$S^* = \cos(\tfrac{\pi}{n})S_{MB} - \sin(\tfrac{\pi}{n})C_{MB},$$

where C_{MB} and S_{MB} are defined by 2.1.5. Hence

$$C^{*2}+S^{*2} = (\cos(\tfrac{\pi}{n})C_{MB} + \sin(\tfrac{\pi}{n})S_{MB})^2 + (\cos(\tfrac{\pi}{n})S_{MB} - \sin(\tfrac{\pi}{n})C_{MB})^2$$

$$= \cos^2(\tfrac{\pi}{n})(C_{MB}^2+S_{MB}^2) + \sin^2(\tfrac{\pi}{n})(C_{MB}^2+S_{MB}^2)$$

$$= C_{MB}^2 + S_{MB}^2.$$

Clearly, further clockwise rotations of the observations will not affect the value of MB. Since any counterclockwise rotation of the observations can be obtained by an appropriately chosen clockwise rotation, we see that the invariance of MB with respect to clockwise or counterclockwise rotations of the observations is verified. □

Lemma 2.2.5. MB is invariant with respect to reflections of the observations about the horizontal axis.

Proof. A reflection about the horizontal axis results in a reversal of the ordering of the projected angles. If $\phi_1^*,\ldots,\phi_n^*$ denote

the projected angles of the reflected vectors, and $\phi_1^*,\ldots,\phi_n^*$ denote again the projected angles of the original vectors, then, for all i,

$$\text{Rank}(\phi_i^*) = n + 1 - \text{Rank}(\phi_i).$$

Although the standardized distances, D_i, are not altered, it is easily seen that each of the sign functions change. The modification of Blumen's statistic applied now to the reflected vectors is $6n^{-1}(C^{*2}+S^{*2})$, where here

$$C^* = \frac{1}{n}\sum_{i=1}^{n} -a_i\cos\left(\frac{\pi(n+1-\text{Rank}(\phi_i))}{n}\right)\text{Rank}(D_i)$$

and

$$S^* = \frac{1}{n}\sum_{i=1}^{n} -a_i\sin\left(\frac{\pi(n+1-\text{Rank}(\phi_i))}{n}\right)\text{Rank}(D_i).$$

Again making use of the trigonometric identities given in 2.2.4, we write

$$C^* = -\frac{1}{n}\sum_{i=1}^{n} a_i\left\{\cos\left(\frac{\pi(n+1)}{n}\right)\cos\left(\frac{\pi\text{Rank}(\phi_i)}{n}\right)+\sin\left(\frac{\pi(n+1)}{n}\right)\sin\left(\frac{\pi\text{Rank}(\phi_i)}{n}\right)\right\}\text{Rank}(D_i)$$

$$= -\left\{\cos\left(\frac{\pi(n+1)}{n}\right)C_{MB} + \sin\left(\frac{\pi(n+1)}{n}\right)S_{MB}\right\}$$

and

$$S^* = -\frac{1}{n}\sum_{i=1}^{n} a_i\{\sin(\frac{\pi(n+1)}{n})\cos(\frac{\pi Rank(\phi_i)}{n}) - \cos(\frac{\pi(n+1)}{n})\sin(\frac{\pi Rank(\phi_i)}{n})\}Rank(D_i)$$

$$= -\{\sin(\frac{\pi(n+1)}{n})C_{MB} - \cos(\frac{\pi(n+1)}{n})S_{MB}\},$$

so that

$$C^{*2} + S^{*2} = C_{MB}^2 + S_{MB}^2. \quad \square$$

Lemma 2.2.6. MB is invariant with respect to changes in scale.

Proof. Consider, without loss in generality, multiplying the first element of each bivariate observation by the positive constant, b. To produce this scale change we multiply each observation by the 2×2 positive definite matrix.

$$S = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}.$$

As illustrated by 2.2.2, the standardized distances, D_i, remain unaffected. It is clear that the sign functions are also unaffected as each observation cannot be transformed across the horizontal axis. The ordering of the angles is unaffected as well, as can be seen by considering the effect of a change in scale on the slopes of the observed vectors. The slopes of the transformed vectors will all be a constant positive multiple of the slopes of the associated original vectors. Trivially, MB is thus unaffected. $\square$

Theorem 2.2.7. MB is invariant with respect to nonsingular linear transformations of the observations.

Proof. Let us represent a nonsingular linear transformation by a 2×2 nonsingular matrix, L. This linear transformation applied to $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ results in the transformed observations $L\underset{\sim}{Y}_1,\ldots,L\underset{\sim}{Y}_n$. Using the singular value decomposition theorem (see, e.g., Rao, 1973, p. 42), we can represent L as $P\Delta Q'$, where P and Q are 2×2 orthogonal matrices and Δ is a 2×2 diagonal matrix with positive diagonal elements. Hence, it suffices to show that MB is invariant with respect to the transformations defined by P, Δ, and Q. That is, we need to show that $MB(\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n) = MB(P\underset{\sim}{Y}_1,\ldots,P\underset{\sim}{Y}_n) = MB(\Delta\underset{\sim}{Y}_1,\ldots,\Delta\underset{\sim}{Y}_n)$, for any 2×2 orthogonal matrix P and any 2×2 diagonal matrix Δ with positive diagonal elements. Firstly, note that the transformation, Δ, produces a change in scale as discussed in Lemma 2.2.6. It remains, therefore, to consider only orthogonal transformations. A 2×2 orthogonal matrix, P, can be represented by either

$$P_1 = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

or

$$P_2 = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix},$$

for some β between 0 and 2π. It is easily established that $P_1\underset{\sim}{Y}$ is a clockwise rotation of $\underset{\sim}{Y}$ through an angle of β radians. Upon noting that

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix},$$

we see that $P_2\underset{\sim}{Y}$ involves a clockwise rotation of $\underset{\sim}{Y}$ through an angle of β followed by a reflection about the horizontal axis. Hence, by Lemmas 2.2.1 and 2.2.5, the invariance of MB with respect to nonsingular linear transformations is established. □

The effect of a nonsingular linear transformation, on the observations themselves, has already been illustrated through the three lemmas. It may be worth noting again that such a transformation amounts to a change in the variance-covariance structure of the distribution of the observed random vectors. Because of the invariance property of MB, we will later be able to restrict attention to simplified variance-covariance structures. Note finally that the preceding results apply to Blumen's statistic as well. That is, we have simultaneously demonstrated the invariance of Blumen's statistic to nonsingular linear transformations.

CHAPTER 3
ASYMPTOTICS FOR THE ONE-SAMPLE PROBLEM

3.1 Asymptotic Null Distribution of MB

In this section we will find the asymptotic null distribution of MB under a class of elliptically symmetric distributions. We first express MB in a more convenient fashion. Let

$$H_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(\phi_i \leq t)$$

and

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(D_i \leq t)$$

represent the empirical distribution functions of the projected angles $\phi_1,\ldots,\phi_n$ and the standardized distances $D_1,\ldots,D_n$, respectively. We can rewrite the expressions for C_{MB} and S_{MB} given in 2.1.5 as

$$C_{MB} = \sum_{i=1}^{n} a_i \cos(\pi H_n(\phi_i)) F_n(D_i)$$

and (3.1.1)

$$S_{MB} = \sum_{i=1}^{n} a_i \sin(\pi H_n(\phi_i)) F_n(D_i).$$

In finding the asymptotic null distribution of $MB = 6n^{-1}(C_{MB}^2+S_{MB}^2)$, we construct a random vector which has the same asymptotic null distribution as $n^{-1/2}(C_{MB}, S_{MB})$. Let, for $i = 1,\ldots,n$,

$$\Sigma_o = E[\underset{\sim}{Y}_i\underset{\sim}{Y}_i'],$$

$$D_i^* = \underset{\sim}{Y}_i' \Sigma_o^{-1} \underset{\sim}{Y}_i,$$

$$H(t) = P_{H_o}(\phi_i \leq t),$$

and

$$F(t) = P_{H_o}(D_i^* \leq t).$$

Define

$$\bar{C}_{MB} = \sum_{i=1}^{n} a_i \cos(\pi H(\phi_i))F(D_i^*)$$

and (3.1.2)

$$\bar{S}_{MB} = \sum_{i=1}^{n} a_i \sin(\pi H(\phi_i))F(D_i^*).$$

It will be shown that $n^{-1/2}(C_{MB}, S_{MB})$ and $n^{-1/2}(\bar{C}_{MB}, \bar{S}_{MB})$ have the same asymptotic null distribution under certain conditions. To verify this we will need to show that, for all i,

$$|H_n(\phi_i) - H(\phi_i)| \overset{H_o}{=} o_p(1)$$

and

$$|F_n(D_i) - F(D_i^*)| \overset{H_o}{=} o_p(1).$$

These conditions will be shown to hold using the following two lemmas.

Lemma 3.1.3. Suppose $X_{n1},\ldots,X_{nn}$ are independent random variables with distribution functions $F_{n1},\ldots,F_{nn}$, respectively, and empirical distribution function $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I[X_{ni} \le t]$. Let $\bar{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} F_{ni}(t)$, then

$$\sup_{-\infty<t<\infty} |F_n(t) - \bar{F}_n(t)| \to 0 \quad \text{a.s., as } n \to \infty.$$

Proof. See Theorem 1 of Shorack and Wellner (1986, p. 106). (If $F_{n1} = \ldots = F_{nn} = F$, then this result reduces to the more familiar result of Glivenko and Cantelli. See, e.g., Serfling, 1980, p. 61.)

Lemma 3.1.4. Let $X_1,\ldots,X_n$ be independent and identically distributed as X, where X has distribution function $F_{\underset{\sim}{\alpha}}$, $\underset{\sim}{\alpha} = (\alpha_1,\ldots,\alpha_p)'$. Assume $g(X,\underset{\sim}{\alpha})$ is a real valued function with $P(g(X,\underset{\sim}{\alpha}) \le t) = G_{\underset{\sim}{\alpha}}(t)$ and that $G_{\underset{\sim}{\alpha}}(t)$ is continuous in t. Let $\hat{\underset{\sim}{\alpha}}_n = (\hat{\alpha}_{n1},\ldots,\hat{\alpha}_{np})'$ be an estimator of $\underset{\sim}{\alpha}$. Suppose that

$$\text{(i)} \quad n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) = O_p(1), \text{ as } n \to \infty,$$

and (3.1.5)

$$\text{(ii)} \quad \sup_{\underset{\sim}{s}\in\beta} |g(X,\underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) - g(X,\underset{\sim}{\alpha})| = o_p(1), \text{ as } n \to \infty,$$

for any bounded sphere β in R^p. Let

$$G_{n,\hat{\underset{\sim}{\alpha}}_n}(t) = \frac{1}{n}\sum_{i=1}^{n} I[g(X_i, \hat{\underset{\sim}{\alpha}}_n) \le t]$$

denote the empirical distribution function of the $g(X_i, \hat{\underset{\sim}{\alpha}}_n)$'s. Then for every positive integer i,

$$G_{n,\hat{\underset{\sim}{\alpha}}_n}(g(X_i, \hat{\underset{\sim}{\alpha}}_n)) - G_{\underset{\sim}{\alpha}}(g(X_i, \underset{\sim}{\alpha})) = o_p(1), \tag{3.1.6}$$

as $n \to \infty$.

Proof. First we will show that conditions (i) and (ii) imply $g(X,\hat{\underset{\sim}{\alpha}}_n) - g(X, \underset{\sim}{\alpha}) = o_p(1)$. Note by (i) that for any $\delta > 0$, there exists a bounded sphere, β, such that

$$P(n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \notin \beta) < \delta, \text{ for all } n. \tag{3.1.7}$$

Let $\delta > 0$, $\delta' > 0$, and $\varepsilon > 0$ be arbitrary. Write

$$\begin{aligned} &P(|g(X,\hat{\underset{\sim}{\alpha}}_n) - g(X,\underset{\sim}{\alpha})| > \varepsilon) \\ &\quad = P(|g(X,\hat{\underset{\sim}{\alpha}}_n) - g(X,\underset{\sim}{\alpha})| > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta) \\ &\qquad + P(|g(X,\hat{\underset{\sim}{\alpha}}_n) - g(X,\underset{\sim}{\alpha})| > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \notin \beta) \\ &\quad \le P(|g(X,\hat{\underset{\sim}{\alpha}}_n) - g(X,\underset{\sim}{\alpha})| > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta) \\ &\qquad + \delta. \end{aligned}$$

Letting

$$W_n(\underset{\sim}{s}) = g(X, \underset{\sim}{\alpha} + n^{-1/2}\underset{\sim}{s}) - g(X, \underset{\sim}{\alpha})$$

so that

$$W_n(n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha})) = g(X, \hat{\underset{\sim}{\alpha}}_n) - g(X, \underset{\sim}{\alpha}),$$

we have

$$P(|g(X,\hat{\underset{\sim}{\alpha}}_n) - g(X,\underset{\sim}{\alpha})| > \varepsilon)$$

$$\leq P(|W_n(n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}))| > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta) + \delta$$

$$\leq P(\sup_{\underset{\sim}{s}\in\beta} |W_n(\underset{\sim}{s})| > \varepsilon) + \delta$$

$$\leq \delta' + \delta, \text{ for large } n,$$

by condition (ii). Using this result, we will complete the proof. Let $\delta > 0$ and $\varepsilon > 0$ be arbitrary. Write

$$|G_{n,\hat{\underset{\sim}{\alpha}}_n}(g(X_i, \hat{\underset{\sim}{\alpha}}_n)) - G_{\underset{\sim}{\alpha}}(g(X_i, \underset{\sim}{\alpha}))|$$

$$\leq |G_{n,\hat{\underset{\sim}{\alpha}}_n}(g(X_i, \hat{\underset{\sim}{\alpha}}_n)) - G_{\underset{\sim}{\alpha}}(g(X_i, \hat{\underset{\sim}{\alpha}}_n))|$$

$$+ |G_{\underset{\sim}{\alpha}}(g(X_i, \hat{\underset{\sim}{\alpha}}_n)) - G_{\underset{\sim}{\alpha}}(g(X_i, \underset{\sim}{\alpha}))|.$$

Using the fact that $g(X, \hat{\underset{\sim}{\alpha}}_n) - g(X,\underset{\sim}{\alpha}) = o_p(1)$ and $G_{\underset{\sim}{\alpha}}(\cdot)$ is continuous, we see that the second term in the sum converges in probability to zero. We need only show that

$$|G_{n,\hat{\underset{\sim}{\alpha}}_n}(g(X_i, \hat{\underset{\sim}{\alpha}}_n)) - G_{\underset{\sim}{\alpha}}(g(X_i, \hat{\underset{\sim}{\alpha}}_n))|$$

also converges in probability to zero. Let

$$G_{n,\underset{\sim}{\alpha}^*}(t) = \frac{1}{n}\sum_{i=1}^{n} I[g(X_i, \underset{\sim}{\alpha}^*) \le t], \text{ for } \underset{\sim}{\alpha}^* \in R^p,$$

$$T_n(\underset{\sim}{s}) = \sup_{-\infty<t<\infty} |G_{n,\underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}}(t) - G_{\underset{\sim}{\alpha}}(t)|,$$

and

$$Q_n = T_n(n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha})).$$

It suffices then to show that $Q_n \xrightarrow{P} 0$. Now

$$P(Q_n > \varepsilon)$$

$$= P(Q_n > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta) + P(Q_n > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \notin \beta)$$

$$\le P(Q_n > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta) + \delta,$$

by 3.1.7. Also,

$$P(Q_n > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta)$$

$$= P(T_n(n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha})) > \varepsilon,\ n^{1/2}(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{\alpha}) \in \beta)$$

$$\le P(\sup_{\underset{\sim}{s}\in\beta} T_n(\underset{\sim}{s}) > \varepsilon).$$

So, we can conclude our proof by showing $\sup_{\underset{\sim}{s}\in\beta} T_n(\underset{\sim}{s}) \xrightarrow{P} 0$. Recalling that

$$G_{n,\underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}}(t) = \frac{1}{n}\sum_{i=1}^{n} I[g(X_i, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \le t],$$

we have for $\underset{\sim}{s} \in \beta$,

$$L(t) \le G_{n,\underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}}(t) \le U(t),$$

where

$$L(t) = \frac{1}{n}\sum_{i=1}^{n} I[\sup_{\underset{\sim}{s}\in\beta} g(X_i, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \le t]$$

and

$$U(t) = \frac{1}{n}\sum_{i=1}^{n} I[\inf_{\underset{\sim}{s}\in\beta} g(X_i, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \le t].$$

Hence, we can bound

$$\sup_{\underset{\sim}{s}\in\beta} \left|G_{n,\underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}}(t) - G_{\underset{\sim}{\alpha}}(t)\right|$$

by (3.1.8)

$$\max\{\left|L(t) - G_{\underset{\sim}{\alpha}}(t)\right|, \left|U(t) - G_{\underset{\sim}{\alpha}}(t)\right|\}.$$

Finally, we have

$$\sup_{-\infty<t<\infty} |L(t) - G_{\underset{\sim}{\alpha}}(t)|$$

$$\leq \sup_{-\infty<t<\infty} |L(t) - P(\sup_{s\in\beta} g(X, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \leq t)|$$

$$+ \sup_{-\infty<t<\infty} |P(\sup_{s\in\beta} g(X, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \leq t) - G_{\underset{\sim}{\alpha}}(t)|.$$

The first term in the above sum converges to zero in probability by lemma 3.1.3, noting that the distribution functions denoted in the lemma as $F_{n1},\ldots,F_{nn}$ are identical and equal to the distribution function of $\sup_{s\in\beta} g(X, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s})$, here. Since, by ii,

$$\sup_{s\in\beta} g(X, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \xrightarrow{P} g(X, \underset{\sim}{\alpha})$$

and $G_{\underset{\sim}{\alpha}}(t) = P(g(X, \underset{\sim}{\alpha}) \leq t)$ is continuous in t,

$$P(\sup_{s\in\beta} g(X, \underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}) \leq t) \to G_{\underset{\sim}{\alpha}}(t),$$

for all t. By Polya's theorem, the convergence is uniform in t. (See, e.g., Serfling, 1980, p. 18.) That is, the second term in the sum above convergences to zero. Treating $\sup_{-\infty<t<\infty} |U(t) - G_{\underset{\sim}{\alpha}}(t)|$ in an analogous fashion, we have, by 3.1.8, that

$$\sup_{s\in\beta} T_n(\underset{\sim}{s}) = \sup_{s\in\beta} \sup_{-\infty<t<\infty} |G_{n,\underset{\sim}{\alpha}+n^{-1/2}\underset{\sim}{s}}(t) - G_{\underset{\sim}{\alpha}}(t)| = o_p(1). \quad \square$$

In our problem conditions (i) and (ii), of lemma 3.1.4 take the form

$$(i') \quad n^{1/2}(\bar{\Sigma}^{-1} - \Sigma_o^{-1}) \overset{H_o}{=} o_p(1), \text{ as } n \to \infty$$

and (3.1.9)

$$(ii') \quad \sup_{s\in\beta} |\underset{\sim}{Y}'(\Sigma_o^{-1} + n^{-1/2}s)\underset{\sim}{Y} - \underset{\sim}{Y}'\Sigma_o^{-1}\underset{\sim}{Y}| \overset{H_o}{=} o_p(1),$$

as $n \to \infty$, where recall $\bar{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{Y}_i\underset{\sim}{Y}_i'$, $\Sigma_o = E[\underset{\sim}{Y}\underset{\sim}{Y}']$, and here β is a set of 2×2 symmetric matrices whose elements are uniformly bounded. We establish condition (i') by showing that each element of $n^{1/2}(\bar{\Sigma}^{-1} - \Sigma_o^{-1})$ is asymptotically normally distributed. The following lemma, found in Serfling (1980, p. 122), will be used.

Lemma 3.1.10. Suppose that $\underset{\sim}{X}_n = (X_{n1},\ldots,X_{nk})'$, $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_k)'$, and $n^{1/2}(\underset{\sim}{X}_n - \underset{\sim}{\mu})$ is asymptotically K-variate normal with mean $\underset{\sim}{0}$ and covariance matrix Σ. Let $g(\underset{\sim}{X}) = (g_1(\underset{\sim}{X}),\ldots,g_m(\underset{\sim}{X}))$ be a vector valued function for which each component $g_i(\underset{\sim}{X})$, $i = 1,\ldots,m$, is real-valued and has a nonzero differential at $\underset{\sim}{X} = \underset{\sim}{\mu}$. Then, $n^{1/2}(g(\underset{\sim}{X}_n) - g(\underset{\sim}{\mu}))$ is asymptotically m-variate normal with mean vector $\underset{\sim}{0}$ and covariance matrix $D\Sigma D'$, where the (i, j) element of D is

$$D_{i,j} = \frac{\partial g_i(\underset{\sim}{X})}{\partial X_j}\Big|_{\underset{\sim}{X} = \underset{\sim}{\mu}}.$$

Remark 3.1.11. A sufficient condition for g_i to have a nonzero differential at $\underset{\sim}{\mu}$ is that the first partial derivatives $\frac{\partial g_i}{\partial X_j}$,

$1 \le j \le K$, are continuous and not all zero at $\underset{\sim}{\mu}$. (See Apostol, 1957, pp. 110 and 118.)

We now prove that conditions (i′) and (ii′) hold.

Lemma 3.1.12. Let $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ be i.i.d. as $\underset{\sim}{Y}$, where $\underset{\sim}{Y} = (Y_1, Y_2)'$ has a bivariate continuous distribution such that the covariance matrix of (Y_1^2, Y_1Y_2, Y_2^2) is positive definite. Then,

$$n^{1/2}(\tilde{\Sigma}^{-1} - \Sigma_o^{-1}) = O_p(1), \text{ as } n \to \infty$$

and

$$\sup_{s\in\beta} \left| \underset{\sim}{Y}'(\Sigma_o^{-1} + n^{-1/2}s)\underset{\sim}{Y} - \underset{\sim}{Y}'\Sigma_o^{-1}\underset{\sim}{Y} \right| = o_p(1),$$

as $n \to \infty$, where $\tilde{\Sigma}$, Σ_o, and β are as in 3.1.9.

Proof. Write

$$\tilde{\Sigma} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \sum_{i=1}^{n} Y_{1i}^2 & \sum_{i=1}^{n} Y_{1i}Y_{2i} \\ \sum_{i=1}^{n} Y_{1i}Y_{2i} & \sum_{i=1}^{n} Y_{2i}^2 \end{pmatrix}$$

and

$$\Sigma_o = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} E[Y_{1i}^2] & E[Y_{1i}Y_{2i}] \\ E[Y_{1i}Y_{2i}] & E[Y_{2i}^2] \end{pmatrix}.$$

Then we can write

$$\tilde{\Sigma}^{-1} = \begin{pmatrix} \dfrac{r_{22}}{r_{11}r_{22}-r_{12}^2} & \dfrac{-r_{12}}{r_{11}r_{22}-r_{12}^2} \\ \dfrac{-r_{12}}{r_{11}r_{22}-r_{12}^2} & \dfrac{r_{11}}{r_{11}r_{22}-r_{12}^2} \end{pmatrix}$$

and

$$\Sigma_o^{-1} = \begin{pmatrix} \frac{\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} & \frac{-\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \\ \frac{-\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} & \frac{\sigma_{11}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \end{pmatrix}.$$

By the multivariate version of the Lindeberg-Levy Central Limit Theorem (see, e.g., Serfling, 1980, p. 28),

$$n^{1/2}(\underset{\sim}{r}-\underset{\sim}{\sigma}) \underset{H_o}{\overset{d}{\rightarrow}} N_3(\underset{\sim}{0}, W), \tag{3.1.13}$$

where

$$\underset{\sim}{r} = (r_{11}, r_{22}, r_{12})', \quad \underset{\sim}{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{12})',$$

$$W = E[((Y_{1i}^2, Y_{2i}^2, Y_{1i}Y_{2i})-\underset{\sim}{\sigma}')'((Y_{1i}^2, Y_{2i}^2, Y_{1i}Y_{2i})-\underset{\sim}{\sigma}')],$$

and $N_3(\underset{\sim}{0}, W)$ refers to the three dimensional normal distribution with mean vector $\underset{\sim}{0}$ and covariance matrix W. We now show that $n^{1/2}(r_{ij}^*-\sigma_{ij}^*)$ is asymptotically normal where r_{ij}^* and σ_{ij}^* refer to the (i,j) elements of $\tilde{\Sigma}^{-1}$ and Σ_o^{-1}, respectively, $1 \leq i \leq j \leq 2$. Taking $i = j = 1$, define $g(\underset{\sim}{r}) = \frac{r_{22}}{r_{11}r_{22}-r_{12}^2}$. Then

$$\frac{\partial g}{\partial r_{11}}\Big|_{\underset{\sim}{r}=\underset{\sim}{\sigma}} = \frac{-r_{22}^2}{(r_{11}r_{22}-r_{12}^2)^2}\Big|_{\underset{\sim}{r}=\underset{\sim}{\sigma}} = \frac{-\sigma_{22}^2}{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)^2} \neq 0,$$

$$\frac{\partial g}{\partial r_{22}}\Big|_{\underset{\sim}{r}=\underset{\sim}{\sigma}} = \frac{-\sigma_{12}^2}{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)^2},$$

and

$$\frac{\partial g}{\partial r_{12}}\Big|_{\underset{\sim}{r}=\underset{\sim}{\sigma}} = \frac{2\sigma_{12}\sigma_{22}}{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)^2}.$$

By lemma 3.1.10, remark 3.1.11, and 3.1.13, $n^{1/2}(r_{11}^*-\sigma_{11}^*)$ is asymptotically normal and hence bounded in probability. In the same manner, it is easily shown that $n^{1/2}(r_{12}^*-\sigma_{12}^*)$ and $n^{1/2}(r_{22}^*-\sigma_{22}^*)$ are also both bounded in probability.

In verifying the second result of the lemma, we let $M > 0$ be the bound on each element of the matrices in β. Then for $s \in \beta$,

$$\begin{aligned}
&|\underset{\sim}{Y}'(\underset{\sim}{\Sigma}_o^{-1}+n^{-1/2}s)\underset{\sim}{Y} - \underset{\sim}{Y}'\underset{\sim}{\Sigma}_o^{-1}\underset{\sim}{Y}| \\
&\quad = |\underset{\sim}{Y}'n^{-1/2}s\underset{\sim}{Y}| = n^{-1/2}|Y_1^2s_{11}+Y_2^2s_{22}+2Y_1Y_2s_{12}| \\
&\quad \le n^{-1/2}M(Y_1^2+Y_2^2+2|Y_1Y_2|),
\end{aligned}$$

where

$$s = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}.$$

That is,

$$\sup_{s\in\beta} |\underset{\sim}{Y}'(\Sigma_o^{-1}+n^{-1/2}s)\underset{\sim}{Y} - \underset{\sim}{Y}'\Sigma_o^{-1}\underset{\sim}{Y}|$$

$$\leq n^{-1/2}M(Y_1^2+Y_2^2+2|Y_1Y_2|) \xrightarrow{P} 0,$$

as $n \to \infty$. □

We are now prepared to show that $n^{-1/2}(C_{MB}, S_{MB})$ and $n^{-1/2}(\bar{C}_{MB}, \bar{S}_{MB})$ have the same asymptotic distribution under H_o. As mentioned, we are restricting attention to the class of elliptically symmetric distributions. The density function of an elliptically symmetric bivariate random vector, $\underset{\sim}{Y}$, is of the form

$$f_{\underset{\sim}{\theta}}(\underset{\sim}{y}) = K|\Sigma|^{-1/2}g((\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})), \; y \in R^2, \qquad (3.1.14)$$

where g is a real valued function, K is a constant, $\underset{\sim}{\theta} = E[\underset{\sim}{Y}]$, if it exists, and Σ is proportional to the covariance matrix of $\underset{\sim}{Y}$, if it exists. We will assume for simplicity that Σ is actually equal to the covariance matrix of $\underset{\sim}{Y}$. For this class of densities we have the following result.

Lemma 3.1.15. Let $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be independently and identically distributed as $\underset{\sim}{Y}$, where $\underset{\sim}{Y}$ is an elliptically symmetric bivariate vector with mean $\underset{\sim}{\theta}$ and positive definite covariance matrix Σ. Assume also that the covariance matrix of (Y_1^2, Y_1Y_2, Y_2^2) is positive definite. Then, under H_o: $\underset{\sim}{\theta} = \underset{\sim}{0}$,

$$n^{-1/2}(C_{MB}-\bar{C}_{MB}, S_{MB}-\bar{S}_{MB}) \xrightarrow{P} (0, 0), \text{ as } n \to \infty.$$

Proof. Consider first

$$E_{H_o}[(n^{-1/2}(C_{MB}-\bar{C}_{MB}))^2]$$

$$= E_{H_o}[n^{-1}\{\sum_{i=1}^{n} a_i(\cos(\pi H_n(\phi_i))F_n(D_i) - \cos(\pi H(\phi_i))F(D_i^*))\}^2]$$

$$= E_{H_o}[n^{-1}\sum_{i=1}^{n} a_i^2(\cos(\pi H_n(\phi_i))F_n(D_i) - \cos(\pi H(\phi_i))F(D_i^*))^2]$$

$$+ E_{H_o}[n^{-1}\sum_{i\neq j} a_i a_j(\cos(\pi H_n(\phi_i))F_n(D_i) - \cos(\pi H(\phi_i))F(D_i^*))$$

$$\times (\cos(\pi H_n(\phi_j))F_n(D_j) - \cos(\pi H(\phi_j))F(D_j^*))].$$

(3.1.16)

Since $\underset{\sim}{Y}$ is elliptically symmetric, $\underset{\sim}{Y}$ and $-\underset{\sim}{Y}$ have the same likelihood under H_o. If we let $\underset{\sim}{Y}_1^P,\ldots,\underset{\sim}{Y}_n^P$ denote, respectively, the projections of the vectors $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ above the horizontal axis, then under H_o $a_1,\ldots,a_n$ are independent of the unsigned projected vectors, $\underset{\sim}{Y}_1^P,\ldots,\underset{\sim}{Y}_n^P$. It follows that under H_o, $a_1,\ldots,a_n$ are independent of the projected angles $\phi_1,\ldots,\phi_n$ and the standardized distances $D_1,\ldots,D_n$ and $D_1^*,\ldots,D_n^*$, since these are all functions of $\underset{\sim}{Y}_1^P,\ldots,\underset{\sim}{Y}_n^P$ only. Therefore, $a_1,\ldots,a_n$ are under H_o i.i.d. with $P(a_i=1) = P(a_i=-1) = 1/2$, $E_{H_o}[a_i] = 0$, and the expectation given by the second term in the sum on the right hand side of 3.1.16 is zero. The first term in that sum is equal to

$$E_{H_o}[\{\cos(\pi H_n(\phi_i))F_n(D_i) - \cos(\pi H(\phi_i))F(D_i^*)\}^2]. \qquad (3.1.17)$$

We will show that, for all $1 \leq i \leq n$,

$$\cos(\pi H_n(\phi_i))F_n(D_i) - \cos(\pi H(\phi_i))F(D_i^*) = o_p(1), \qquad (3.1.18)$$

as $n \to \infty$. We then apply Lebesgue Dominated Convergence Theorem (noting that the expression is 3.1.17 is bounded in magnitude by 4) to show that 3.1.17 and hence 3.1.16 converge to zero, as $n \to \infty$. Clearly, it suffices then to show that

$$|H_n(\phi_i) - H(\phi_i)| = o_p(1)$$

and (3.1.19)

$$|F_n(D_i) - F(D_i^*)| = o_p(1),$$

as $n \to \infty$. Lemma 3.1.3 (or Glivenko-Cantelli) establishes immediately that

$$\sup_{-\infty<t<\infty} |H_n(t) - H(t)| = o_p(1), \text{ as } n \to \infty, \qquad (3.1.20)$$

taking $F_{n1},\ldots,F_{nn}$ of the lemma to each be equal to H. Showing $|F_n(D_i) - F(D_i^*)| = o_p(1)$, as $n \to \infty$, requires the use of lemma 3.1.4. Conditions (i) and (ii) of Lemma 3.1.4 are precisely those established in Lemma 3.1.12. This establishes that the expression in 3.1.17 converges to zero, as $n \to \infty$, and hence, by 3.1.16, $E_{H_o}[\{n^{-1/2}(C_{MB}-\bar{C}_{MB})\}^2] = o_p(1)$, as $n \to \infty$. By the Markov inequality then, under H_o $n^{-1/2}(C_{MB}-\bar{C}_{MB}) = o_p(1)$. In the same fashion it can be shown that $n^{1/2}(S_{MB}-\bar{S}_{MB}) = o_p(1)$, under H_o. □

The asymptotic null distribution of $n^{-1/2}(C_{MB}, S_{MB})$ can now be found by finding the asymptotic null distribution of $n^{-1/2}(\bar{C}_{MB}, \bar{S}_{MB})$.

Theorem 3.1.21. If the conditions of Lemma 3.1.15 hold, then under H_o: $\underset{\sim}{\theta} = \underset{\sim}{0}$, $MB = 6n^{-1}(C_{MB}^2+S_{MB}^2)$ is asymptotically distributed as a Chi-square random variable with two degrees of freedom.

Proof. We find the asymptotic null distribution of $(6n^{-1})^{1/2}(\bar{C}_{MB}, \bar{S}_{MB})$. Let us write

$$\bar{C}_{MB} = \sum_{i=1}^{n} \bar{C}_i \quad \text{and} \quad \bar{S}_{MB} = \sum_{i=1}^{n} \bar{S}_i,$$

where (3.1.22)

$$\bar{C}_i = a_i\cos(\pi H(\phi_i))F(D_i^*) \quad \text{and} \quad \bar{S}_i = a_i\sin(\pi H(\phi_i))F(D_i^*),$$

$i = 1,\ldots,n$. By the multivariate version of the Lindeberg-Levy Central Limit Theorem, under H_o,

$$\left(\frac{\bar{C}_{MB}-nE_{H_o}[\bar{C}_i]}{\{nVar_{H_o}[\bar{C}_i]\}^{1/2}} \;,\; \frac{\bar{S}_{MB}-nE_{H_o}[\bar{S}_i]}{\{nVar_{H_o}[\bar{S}_i]\}^{1/2}}\right) \tag{3.1.23}$$

is asymptotically normal with mean vector $\underset{\sim}{0}$ and covariance matrix

$$\Gamma = \begin{pmatrix} 1 & \text{correlation}[\bar{C}_i, \bar{S}_i] \\ \text{correlation}[\bar{C}_i, \bar{S}_i] & 1 \end{pmatrix}. \tag{3.1.24}$$

Since $E_{H_o}[a_i] = 0$ and a_i is independent of both ϕ_i and D_i^*, $E_{H_o}[\bar{C}_i] = E_{H_o}[\bar{S}_i] = 0$. Using the form of the density in 3.1.14, a simple change of variable argument will show that ϕ_i and D_i^* are independent under H_o, for $i = 1,\ldots,n$. Hence, for all $1 \leq i \leq n$,

$$\begin{aligned} \mathrm{Var}_{H_o}[\bar{C}_i] &= E_{H_o}[a_i^2\cos^2(\pi H(\phi_i))F^2(D_i^*)] \\ &= E_{H_o}[\cos^2(\pi H(\phi_i))]E_{H_o}[F^2(D_i^*)] \end{aligned}$$

and (3.1.25)

$$\mathrm{Var}_{H_o}[\bar{S}_i] = E_{H_o}[\sin^2(\pi H(\phi_i))]E_{H_o}[F^2(D_i^*)].$$

Under H_o, both $H(\phi_i)$ and $F(D_i^*)$ are uniformly distributed on (0, 1) for all i. Therefore,

$$E_{H_o}[\cos^2(\pi H(\phi_i))] = \int_0^1 \cos^2(\pi u)du = 1/2$$

$$E_{H_o}[\sin^2(\pi H(\phi_i))] = \int_0^1 \sin^2(\pi u)du = 1/2$$

and (3.1.26)

$$E_{H_o}[F^2(D_i^*)] = \int_0^1 u^2 du = 1/3.$$

Also

$$\begin{aligned}
\mathrm{Cov}_{H_o}[\bar{C}_i, \bar{S}_i] &= E_{H_o}[\bar{C}_i \bar{S}_i] \\
&= E_{H_o}[a_i^2 \cos(\pi H(\phi_i))\sin(\pi H(\phi_i))F^2(D_i^*)] \\
&= E_{H_o}[\cos(\pi H(\phi_i))\sin(\pi H(\phi_i))]E_{H_o}[F^2(D_i^*)] \\
&= \int_0^1 \cos(\pi u)\sin(\pi u)du \int_0^1 u^2 du \\
&= 0. \qquad (3.1.27)
\end{aligned}$$

So, using 3.1.23 - 3.1.27, $(6n^{-1})^{1/2}(\bar{C}_{MB}, \bar{S}_{MB})$ is asymptotically bivariate normal under H_o, with mean vector $\underset{\sim}{0}$ and covariance matrix I_2, the identiy matrix of order 2. Lemma 3.1.15 shows that under H_o, $(6n^{-1})^{1/2}(C_{MB}, S_{MB})$ is also asymptotically bivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix I_2. It follows that

$$MB = ((6n^{-1})^{1/2}C_{MB})^2 + ((6n^{-1})^{1/2}S_{MB})^2$$

is asymptotically Chi-square with two degrees of freedom under H_o. □

3.2 Asymptotic Distribution of MB Under Contiguous Alternatives

As the first step in obtaining Pitman asymptotic relative efficiencies, we will find the asymptotic distribution of MB under a sequence of alternatives approaching the null hypothesis. In doing so, we will consider a specific class of elliptically symmetric distributions which has a density function of the form

$$c(\nu)\exp\{\frac{-[(\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})]^{\nu}}{h(\nu)}\} \qquad \text{for } \underset{\sim}{y} \in R^2, \tag{3.2.1}$$

where ν, $c = c(\nu)$ and $h = h(\nu)$ are positive constants. Since MB is invariant with respect to nonsingular linear transformations, we can, without loss of generality, take $\Sigma = I_2$, the identity matrix of order 2. Letting

$$c = \frac{\nu\Gamma(2/\nu)}{2\pi(\Gamma(1/\nu))^2} \quad \text{and} \quad h = \left(\frac{2\Gamma(1/\nu)}{\Gamma(2/\nu)}\right)^{\nu}, \tag{3.2.2}$$

where

$$\Gamma(w) = \int_0^{\infty} u^{w-1}e^{-u}du \qquad \text{for } w > 0,$$

it can be verified that the expression in 3.2.1 is a valid density with covariance matrix equal to I_2. For $\nu = 1$ the density is that of a bivariate normal random vector. Taking $\nu > 1$ and $0 < \nu < 1$ produces densities that are lighter tailed and heavier tailed, respectively, than that of the bivariate normal.

With these underlying distributions we will apply Le Cam's third lemma to find the asymptotic distribution of MB under the alternatives

$$H_n: \quad \underset{\sim}{\theta} = \underset{\sim}{\theta}_n = (0, Kn^{-1/2})'. \tag{3.2.3}$$

(Due to the affine-invariance of MB, we shall see that the form of the alternatives considered results in no loss of generality.)

Lemma 3.2.4 (Le Cam's third lemma; see, e.g., Prakasa Rao, 1987, p. 104). Let μ_n be a measure dominating the measures Q_n and P_n.

Let

$$p_n = \frac{dP_n}{d\mu_n} \quad \text{and} \quad q_n = \frac{dQ_n}{d\mu_n}$$

be the Radon-Nikodym derivatives of P_n and Q_n, respectively, with respect to μ_n. Also let

$$L_n = \begin{cases} q_n/p_n, & \text{if } p_n > 0, \\ 1, & \text{if } q_n = p_n = 0, \\ n, & \text{if } q_n > p_n = 0. \end{cases}$$

Let S_n be a random variable and suppose that under P_n, $(S_n, \log L_n)$ is asymptotically bivariate normal with mean vector $\underset{\sim}{\mu} = (\mu_1, \mu_2)'$ and covariance matrix

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{12} & \Upsilon_{22} \end{pmatrix},$$

where $\mu_2 = -1/2\ \Upsilon_{22}$. Then, under Q_n, S_n converges in distribution to a normal random variable with mean $\mu_1+\Upsilon_{12}$ and variance Υ_{11}.

We will take P_n to be the joint probability function of $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ when $\underset{\sim}{\theta} = \underset{\sim}{0}$, and Q_n, the joint probability function when $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n$. Taking μ_n to be the Lebesgue measure on R^n, it follows that p_n and q_n are the joint density functions associated with $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ under $\underset{\sim}{\theta} = \underset{\sim}{0}$ and $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n$, respectively. S_n will be an arbitrary linear combination of $(6n^{-1})^{1/2}C_{MB}$ and $(6n^{-1})^{1/2}S_{MB}$. Hence, if we can show

that $(S_n, \log L_n)$ is asymptotically bivariate normal under H_o, with parameters satisfying the relationship specified in Le Cam's third lemma, then S_n will be asymptotically normal under the sequence of (contiguous) alternatives. That is, we will have shown that every linear combination of $(6n^{-1})^{1/2}C_{MB}$ and $(6n^{-1})^{1/2}S_{MB}$, is asymptotically normal under H_n, implying that $(6n^{-1})^{1/2}(C_{MB}, S_{MB})$ is itself asymptotically bivariate normal under H_n.

So, let us denote the joint density function of n independent observations from 3.2.1, with $\Sigma = I_2$, by p_n when $\underset{\sim}{\theta} = \underset{\sim}{0}$, and by q_n when $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n = (0, Kn^{-1/2})'$. Let P_n and Q_n denote the corresponding distribution functions. Let

$$L_n = L_n(\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n) = \frac{q_n}{p_n} = \exp\left[\frac{1}{h}\sum_{i=1}^{n}\{[\underset{\sim}{Y}_i'\underset{\sim}{Y}_i]^{\nu} - [(\underset{\sim}{Y}_i - \underset{\sim}{\theta}_n)'(\underset{\sim}{Y}_i - \underset{\sim}{\theta}_n)]^{\nu}\}\right] \tag{3.2.5}$$

Transforming to polar coordinates via $Y_{1i} = Z_i\cos\phi_i'$ and $Y_{2i} = Z_i\sin\phi_i'$, $Z_i > 0$, and letting $\Delta = Kn^{-1/2}$, we may write

$$\log L_n = \frac{1}{h}\sum_{i=1}^{n}\{Z_i^{2\nu} - (Z_i^2 - 2Z_i\sin\phi_i'\Delta + \Delta^2)^{\nu}\}. \tag{3.2.6}$$

We will first find an expression which has the same asymptotic distribution as $\log L_n$ under P_n. To accomplish this we use the following lemma.

Lemma 3.2.7. Suppose for each $n \geq 1$, X_{ni}, $1 \leq i \leq K_n \to \infty$ are independent positive random variables. Then

$$T_n = \sum_{i=1}^{K_n} X_{ni} \xrightarrow{P} 1$$

if for all $\varepsilon > 0$

$$\text{(i)} \quad \sum_{i=1}^{K_n} P(X_{ni} \geq \varepsilon) = o(1)$$

and

$$\text{(ii)} \quad \sum_{i=1}^{K_n} E[X_{ni} I[X_{ni} < 1]] \to 1$$

Proof. See Chow and Teicher (1978, p. 328).

An asymptotic null representation for $\log L_n$ is given by the following lemma.

Lemma 3.2.8. If $\nu > 1/2$, then

$$\log L_n = \frac{K}{\sqrt{n}h} \sum_{i=1}^{n} 2\nu \sin(\phi_i') Z_i^{2\nu-1} - \frac{K^2 W}{h} + o_{P_n}(1), \tag{3.2.9}$$

where

$$W = E_{P_n}[2\nu(\nu-1) Z_i^{2\nu-2} \sin^2(\phi_i') + \nu Z_i^{2\nu-2}]$$
$$= K^2 \nu \pi c. \tag{3.2.10}$$

Proof. Letting

$$h_i(\Delta) = (Z_i^2 - 2Z_i \sin(\phi_i')\Delta + \Delta^2)^{\nu},$$

we have

$$\frac{dh_i(\Delta)}{d\Delta} = \nu(Z_i^2 - 2Z_i\sin(\phi_i')\Delta + \Delta^2)^{\nu-1}(2\Delta - 2Z_i\sin\phi_i'),$$

and

$$\frac{d^2h_i(\Delta)}{d\Delta^2} = \nu(\nu-1)(Z_i^2-2Z_i\sin(\phi_i')\Delta+\Delta^2)^{\nu-2}(2\Delta-2Z_i\sin(\phi_i'))^2$$
$$+ 2\nu(Z_i^2-2Z_i\sin(\phi_i')\Delta+\Delta^2)^{\nu-1}.$$

Expanding $h_i(\Delta)$ about 0 via Taylor's theorem yields

$$h_i(\Delta) = Z_i^{2\nu} - 2\nu\sin(\phi_i')Z_i^{2\nu-1}\Delta + \Delta^2 R_{ni}, \tag{3.2.11}$$

where

$$R_{ni} = 2\nu(\nu-1)(Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2})^{\nu-2}(\Delta_i^*-Z_i\sin(\phi_i'))^2$$
$$+ \nu(Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2})^{\nu-1},$$

for some Δ_i^* between 0 and Δ, $i = 1,\ldots,n$. Hence

$$\log L_n = \frac{1}{h}\sum_{i=1}^{n}(Z_i^{2\nu} - h_i(\Delta))$$
$$= \frac{\Delta}{h}\sum_{i=1}^{n} 2\nu\sin(\phi_i')Z_i^{2\nu-1} - \frac{\Delta^2}{h}\sum_{i=1}^{n} R_{ni}. \tag{3.2.12}$$

We need to show that for $\nu > 1/2$

$$\frac{\Delta^2}{h}\sum_{i=1}^{n} R_{ni} \xrightarrow{P_n} \frac{K^2W}{h}. \tag{3.2.13}$$

Write

$$R_{ni} = \nu(Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2})^{\nu-1} \times [1 - \frac{2(1-\nu)(\Delta_i^*-Z_i\sin(\phi_i'))^2}{(Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2})}]$$

$$= \nu(Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2})^{\nu-1}$$

$$\times [1 - 2(1-\nu)(\frac{\Delta_i^{*2}-2Z_i\sin(\phi_i')\Delta_i^*+Z_i^2\sin^2(\phi_i')}{Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2}})]. \tag{3.2.14}$$

Since

$$0 \le (Z_i^2\sin^2(\phi_i')-2Z_i\sin\phi_i'\Delta_i^*+\Delta_i^{*2}) \le (Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2}), \tag{3.2.15}$$

it follows that for $1/2 \le \nu \le 1$,

$$0 \le R_{ni} \le \nu(Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2})^{\nu-1}$$

$$\le \nu(Z_i^2\cos^2(\phi_i'))^{\nu-1}. \tag{3.2.16}$$

Here we are using the fact that

$$Z_i^2-2Z_i\sin(\phi_i')\Delta_i^*+\Delta_i^{*2} = Y_{1i}^2+(Y_{2i}-\Delta_i^*)^2$$

$$\ge Y_{1i}^2$$

$$= Z_i^2\cos^2(\phi_i').$$

Using 3.2.14 and 3.2.15, if $\nu > 1$ and n is large enough so that $Kn^{-1/2} < 1$, then

$$R_{ni} \leq \nu(Z_i^2+2Z_i+1)^{\nu-1} + 2\nu(\nu-1)(Z_i^2+2Z_i+1)^{\nu-1}. \tag{3.2.17}$$

Using 3.2.16 and 3.2.17, we can for large n bound R_{ni} by

$$B_i(\nu) = \begin{cases} \nu(Z_i^2\cos^2(\phi_i'))^{\nu-1}, & \text{if } 1/2 \leq \nu \leq 1, \\ \nu(2\nu-1)(Z_i^2+2Z_i+1)^{\nu-1}, & \text{if } \nu > 1. \end{cases} \tag{3.2.18}$$

We will show that $B_i(\nu)$ is integrable with respect to P_n. Using again the transformations $Y_{1i} = Z_i\cos(\phi_i')$ and $Y_{2i} = Z_i\sin(\phi_i')$, the joint density of Z_i and ϕ_i' under P_n is, from 3.2.1,

$$f(z_i, \phi_i') = cz_i e^{-\frac{z_i^{2\nu}}{h}} \qquad 0 < z_i < \infty,\ 0 \leq \phi_i' < 2\pi, \tag{3.2.19}$$

where c and h are given in 3.2.2. The marginal densities of Z_i and ϕ_i' under P_n are given, respectively, by

$$g_1(z_i) = 2\pi cz_i e^{-\frac{z_i^{2\nu}}{h}}, \qquad 0 < z_i < \infty$$

and (3.2.20)

$$g_2(\phi_i') = \frac{1}{2\pi}, \qquad 0 \leq \phi_i' < 2\pi.$$

We will be using the fact that Z_i and ϕ_i' are independent, as is readily seen. We find

$$E_{P_n}[Z_i^a] = 2\pi c \int_0^\infty z^{a+1} e^{-\frac{z^{2\nu}}{h}}\, dz$$

$$= \frac{2\pi c}{2\nu} \int_0^\infty x^{\frac{a+2}{2\nu}-1} e^{-\frac{x}{h}}\, dx$$

$$= \frac{\pi c}{\nu} \Gamma\left(\frac{a+2}{2\nu}\right) h^{\frac{a+2}{2\nu}} < \infty, \qquad (3.2.21)$$

if $a > -2$. Also,

$$E_{P_n}[|\cos(\phi_i')|^{2\nu-2}] = \frac{1}{2\pi} \int_0^{2\pi} |\cos u|^{2\nu-2} du$$

$$= \frac{2}{2\pi} \{ \int_0^{\pi/2} (\cos u)^{2\nu-2} du + \int_{\pi/2}^{\pi} (-\cos u)^{2\nu-2} du \}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} (\cos u)^{2\nu-2} du$$

$$= \frac{2}{\pi} \int_0^1 \frac{((1-w)^{1/2})^{2\nu-2}}{2w^{1/2}(1-w)^{1/2}}\, dw \quad \text{(letting } w = \sin^2 u\text{)}$$

$$= \frac{1}{\pi} \int_0^1 (1-w)^{(\nu-\frac{1}{2})-1} w^{\frac{1}{2}-1}\, dw < \infty,$$

(3.2.22)

if $\nu > 1/2$. We see then, using 3.2.18, 3.2.21, and 3.2.22 that $B_i(\nu)$ is integrable with respect to P_n if $\nu > 1/2$.

We will now use Lemma 3.2.7 to show that for $\nu > 1/2$

$$\frac{\Delta^2}{h} \sum_{i=1}^{n} R_{ni} \xrightarrow{P_n} \frac{K^2 W}{h},$$

or equivalently that

$$\frac{\Delta^2}{WK^2} \sum_{i=1}^{n} R_{ni} = \sum_{i=1}^{n} \frac{R_{ni}}{Wn} \xrightarrow{P_n} 1.$$

Taking $Y_{ni} = \frac{R_{ni}}{nW}$, we will first show

$$\sum_{i=1}^{n} P_{P_n}(Y_{ni} \geq \varepsilon) = o(1),$$

for all $\varepsilon > 0$. We have

$$\begin{aligned}
\sum_{i=1}^{n} P_{P_n}(Y_{ni} \geq \varepsilon) &= \sum_{i=1}^{n} P_{P_n}\left(\frac{R_{ni}}{W\varepsilon} \geq n\right) \\
&\leq \sum_{i=1}^{n} P_{P_n}\left(\frac{B_i(\nu)}{W\varepsilon} \geq n\right) \\
&= nP_{P_n}\left(\frac{B_i(\nu)}{W\varepsilon} \geq n\right) \\
&\to 0,
\end{aligned}$$

as $n \to \infty$, since $\frac{B_i(v)}{W\epsilon}$ is integrable. As for condition (ii) of Lemma 3.2.7, write

$$\sum_{i=1}^{n} E_{P_n}[Y_{ni} I[Y_{ni} < 1]] = \sum_{i=1}^{n} E_{P_n}[\frac{R_{ni}}{Wn} I[\frac{R_{ni}}{Wn} < 1]]$$

$$= nE_{P_n}[\frac{R_{ni}}{Wn} I[\frac{R_{ni}}{Wn} < 1]]$$

$$= \frac{1}{W} E_{P_n}[R_{ni} I[\frac{R_{ni}}{Wn} < 1]].$$

Since $R_{ni} \leq B_i(v)$ and $E_{P_n}[B_i(v)] < \infty$, we can use Lebesgue Dominated Convergence Theorem to obtain the result that

$$\lim_{n\to\infty} \sum_{i=1}^{n} E_{P_n}[Y_{ni} I[Y_{ni} < 1]]$$

$$= \lim_{n\to\infty} \frac{1}{W} E_{P_n}[R_{ni} I[\frac{R_{ni}}{nW} < 1]]$$

$$= \frac{1}{W} E_{P_n}[2v(v-1)Z_i^{2v-2}\sin^2(\phi_i') + vZ_i^{2v-2}]$$

$$= 1,$$

where we are using 3.2.11 to find the stochastic limit of R_{ni}, as $n \to \infty$. In conjunction with 3.2.12, this establishes the representation of $\log L_n$ given by 3.2.9. The expectation in 3.2.10 is easily established using 3.2.21 and

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2(\phi_i')d\phi_i' = 1/2. \quad \square$$

The asymptotic P_n distribution of $\log L_n$ easily follows and is given in the next lemma.

Lemma 3.2.23. If $\nu > 1/2$, $\log L_n$ is asymptotically normal under P_n with mean μ and variance -2μ, where $\mu = K^2 W h^{-1}$.

Proof. Let

$$L_n^* = n^{-1/2} \sum_{i=1}^{n} L_i - K^2 W h^{-1},$$

where (3.2.24)

$$L_i = 2\nu K h^{-1} \sin(\phi_i') Z_i^{2\nu-1}, \qquad i = 1,\ldots,n.$$

By the central limit theorem L_n^* is asymptotically normal under P_n with mean

$$E_{P_n}[L_i] - K^2 W h^{-1}$$

and variance (3.2.25)

$$\mathrm{Var}_{P_n}[L_i].$$

We find

$$E_{P_n}[L_i] = 2\nu K h^{-1} E_{P_n}[\sin(\phi_i') Z_i^{2\nu-1}] = 0, \tag{3.2.26}$$

since

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(\phi_i') d\phi_i' = 0,$$

and ϕ_i' and Z_i are independent. Also,

$$\begin{aligned}\mathrm{Var}_{P_n}[L_i] &= E_{P_n}[L_i^2] = (2\nu Kh^{-1})^2 E_{P_n}[\sin^2\phi_i'] E_{P_n}[Z_i^{2(2\nu-1)}] \\ &= (2\nu Kh^{-1})^2 (\tfrac{1}{2})(\tfrac{\pi c}{\nu})\,\Gamma(2)\,h^2) \\ &= 2\nu K^2 \pi c, \end{aligned} \tag{3.2.27}$$

using 3.2.21 and

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2(\phi_i')d\phi_i' = 1/2.$$

By Lemma 3.2.8, $\log L_n$ is under P_n also asymptotically normal with mean $-K^2Wh^{-1}$ and variance $2\nu K^2\pi c$. Furthermore,

$$\begin{aligned}-K^2Wh^{-1} &= -K^2h^{-1}E_{P_n}[2\nu(\nu-1)Z_i^{2\nu-2}\sin^2(\phi_i')+\nu Z_i^{2\nu-2}] \\ &= -K^2h^{-1}E_{P_n}[Z_i^{2\nu-2}]E_{P_n}[2\nu(\nu-1)\sin^2(\phi_i')+\nu] \\ &= (-K^2h^{-1})(\pi c h\nu^{-1})(2\nu(\nu-1)(\tfrac{1}{2})+\nu) \\ &= -K^2\pi c\nu \\ &= -\tfrac{1}{2}(\mathrm{Var}_{P_n}[L_i]). \end{aligned} \tag{3.2.28}$$

This completes the proof and implies that Q_n is contiguous to P_n (see, e.g., Prakasa Rao, 1987, p. 104). □

The following theorem gives the asymptotic distribution of MB under Q_n.

Theorem 3.2.29. Suppose $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independently and identically distributed from a density function given by 3.2.1, with $\nu > 1/2$, and $\Sigma = I_2$. Then, under the alternatives given by 3.2.3, MB is asymptotically distributed as a noncentral Chi-square random variable with two degrees of freedom and noncentrality parameter given by

$$\delta^2_{MB}(\nu) = \left(\frac{\nu K \sqrt{3}\ \Gamma(\frac{3}{2\nu}+1)(\Gamma(\frac{2}{\nu}))^{1/2} \beta_\nu(\frac{1}{2})}{(\Gamma(\frac{1}{\nu}))^{5/2}} \right)^2,$$

where (3.2.30)

$$\Gamma(w) = \int_0^\infty u^{w-1}e^{-u}du,\ w > 0,$$

and

$$\beta_\nu(\tfrac{1}{2}) = \int_0^{\frac{1}{2}} u^{\frac{1}{\nu}-1}(1-u)^{\frac{1}{2\nu}}\,du.$$

Proof. Applying Le Cam's third lemma (Lemma 3.2.4), we need to find the asymptotic distribution of $(S_n, \log L_n)$ under P_n, where

$$S_n = \ell_1(6n^{-1})^{1/2}C_{MB} + \ell_2(6n^{-1})^{1/2}S_{MB}$$

is an arbitrary linear combination of $(6n^{-1})^{1/2}C_{MB}$ and $(6n^{-1})^{1/2}S_{MB}$. By 3.2.24, 3.2.9, and Lemma 3.1.15, the asymptotic distribution of $(S_n, \log L_n)$ under P_n is the same as that of (S_n^*, L_n^*), where

$$S_n^* = \ell_1(6n^{-1})^{1/2}\bar{C}_{MB} + \ell_2(6n^{-1})^{1/2}\bar{S}_{MB}.$$

We will then work with (S_n^*, L_n^*). Let us write

$$\bar{C}_{MB} = \sum_{i=1}^{n} \bar{C}_i \quad \text{and} \quad \bar{S}_{MB} = \sum_{i=1}^{n} \bar{S}_i,$$

where (3.2.31)

$$\bar{C}_i = a_i \cos(\pi H(\phi_i))F(D_i^*)$$

and

$$\bar{S}_i = a_i \sin(\pi H(\phi_i))F(D_i^*).$$

We have already established that

$$E_{P_n}[\bar{C}_i] = E_{P_n}[\bar{S}_i] = 0$$

and

$$\mathrm{Var}_{P_n}[\bar{C}_i] = \mathrm{Var}_{P_n}[\bar{S}_i] = \frac{1}{6}.$$

(See the proof of Theorem 3.1.21.) Therefore, using the multivariate version of the Central Limit Theorem, in conjunction with Lemma 3.2.23, we have that (S_n^*, L_n^*) and hence $(S_n, \log L_n)$ are both asymptotically bivariate normal under P_n with mean $(0, \mu)$ and covariance matrix

$$\Gamma = \begin{pmatrix} \ell_1^2 + \ell_2^2 & \gamma_{12} \\ \gamma_{12} & -2\mu \end{pmatrix}, \qquad (3.2.32)$$

where

$$\gamma_{12} = \mathrm{Cov}_{P_n}[(\ell_1\sqrt{6}\ \bar{C}_i+\ell_2\sqrt{6}\ \bar{S}_i),\ L_i]$$

$$= E_{P_n}[(\ell_1\sqrt{6}\ \bar{C}_i+\ell_2\sqrt{6}\ \bar{S}_i)L_i] \qquad (3.2.33)$$

To find γ_{12}, we need $E_{P_n}[\bar{C}_iL_i]$ and $E_{P_n}[\bar{S}_iL_i]$. Refer to the definitions of ϕ_i', ϕ_i, and D_i^*, given in Chapter 2, and recall that $\Sigma_o = I_2$. Then

$$D_i^* = \underset{\sim}{Y}_i'\Sigma_o^{-1}\underset{\sim}{Y}_i = Y_{1i}^2+Y_{2i}^2 = Z_i^2, \qquad i = 1,\ldots,n,$$

and (3.2.34)

$$F(t^2) = P_{P_n}(D_i^* \le t^2) = P_{P_n}(Z_i^2 \le t^2) = P_{P_n}(Z_i \le t),$$

for $t \ge 0$. Noting that the ϕ_i' defined in conjunction with MB is identical to that used in the preceding lemma, we use 2.1.1 and 3.2.20 to find, for $0 \le t \le \pi$,

$$H(t) = P_{P_n}(\phi_i \le t) = P_{P_n}(0 \le \phi_i' \le t) + P_{P_n}(\pi < \phi_i' \le \pi+t)$$

$$= \frac{t}{2\pi} + \frac{\pi+t}{2\pi} - \frac{\pi}{2\pi}$$

$$= \frac{t}{\pi}\ . \qquad (3.2.35)$$

Letting $G(t) = P_{P_n}(Z_i \le t)$,

$$E_{P_n}[\bar{C}_i L_i] = 2\nu K h^{-1} E_{P_n}[a_i \cos(\pi H(\phi_i)) F(D_i^*) \sin(\phi_i') Z_i^{2\nu-1}]$$

$$= 2\nu K h^{-1} E_{P_n}[a_i \cos(\phi_i)\sin(\phi_i')] E_{P_n}[Z_i^{2\nu-1} G(Z_i)], \qquad (3.2.36)$$

using 3.2.34 and the independence of $\{D_i^*, Z_i\}$ and $\{a_i, \phi_i, \phi_i'\}$, under P_n. Since

$$E_{P_n}[a_i \cos(\phi_i)\sin(\phi_i')]$$

$$= E_{P_n}[\cos(\phi_i')\sin(\phi_i') I[0 \le \phi_i' \le \pi]]$$

$$+ E_{P_n}[-\cos(\phi_i'-\pi)\sin(\phi_i') I[\pi < \phi_i' < 2\pi]]$$

and (3.2.37)

$$\frac{1}{2\pi}\int_0^{\pi} \cos(\phi')\sin(\phi')d\phi' = \frac{1}{2\pi}\int_{\pi}^{2\pi} -\cos(\phi'-\pi)\sin(\phi')d\phi' = 0,$$

we have

$$E_{P_n}[\bar{C}_i L_i] = 0 \qquad (3.2.38)$$

Finally,

$$E_{P_n}[\bar{S}_i L_i] = 2\nu K h^{-1} E_{P_n}[G(Z_i) Z_i^{2\nu-1}] E_{P_n}[a_i \sin(\pi H(\phi_i))\sin(\phi_i')]. \qquad (3.2.39)$$

We find

$$E_{P_n}[a_i \sin(\pi H(\phi_i))\sin(\phi_i')]$$

$$= E_{P_n}[\sin^2(\phi_i')I[0 \le \phi_i' \le \pi]] + E_{P_n}[-\sin(\phi'-\pi)\sin(\phi_i')I[\pi < \phi_i' < 2\pi]]$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin^2(\phi')d(\phi') - \frac{1}{2\pi}\int_{\pi}^{2\pi} \sin(\phi'-\pi)\sin(\phi')d(\phi')$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2(\phi')d(\phi')$$

$$= \frac{1}{2}, \qquad (3.2.40)$$

and using 3.2.20 again,

$$E_{P_n}[G(Z_i)Z_i^{2\nu-1}]$$

$$= 2\pi c \int_0^{\infty} G(z) z^{2\nu} e^{-\frac{z^{2\nu}}{h}} dz$$

$$= 2\pi c \int_0^{\infty} (2\pi c \int_0^{z} y e^{-\frac{y^{2\nu}}{h}} dy) z^{2\nu} e^{-\frac{z^{2\nu}}{h}} dz.$$

Letting $y = x^{\frac{1}{\nu}}\sin^{\frac{1}{\nu}}(u)$ and $z = x^{\frac{1}{\nu}}\cos^{\frac{1}{\nu}}(u)$, we find that

$$\frac{dz}{dx} = \frac{x^{\frac{1}{\nu}-1}\cos^{\frac{1}{\nu}}(u)}{\nu}, \quad \frac{dz}{du} = \frac{-x^{\frac{1}{\nu}}\cos^{\frac{1}{\nu}-1}(u)\sin(u)}{\nu},$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{\nu}-1}\sin^{\frac{1}{\nu}}(u)}{\nu}, \quad \text{and } \frac{dy}{du} = \frac{x^{\frac{1}{\nu}}\sin^{\frac{1}{\nu}-1}(u)\cos(u)}{\nu}.$$

The jacobian of the transformation is

$$\frac{x^{\frac{2}{\nu}-1}}{\nu^2}(\cos^{\frac{1}{\nu}+1}(u)\sin^{\frac{1}{\nu}-1}(u) + \cos^{\frac{1}{\nu}-1}(u)\sin^{\frac{1}{\nu}+1}(u))$$

$$= \frac{x^{\frac{2}{\nu}-1}}{\nu^2}(\cos^{\frac{1}{\nu}-1}(u)\sin^{\frac{1}{\nu}-1}(u))(\cos^2(u)+\sin^2(u))$$

$$= \frac{x^{\frac{2}{\nu}-1}}{\nu^2}(\cos^{\frac{1}{\nu}-1}(u)\sin^{\frac{1}{\nu}-1}(u)).$$

It follows that

$$E_{P_n}[G(Z_i)Z_i^{2\nu-1}]$$

$$= \frac{4\pi^2c^2}{\nu^2}\int_0^{\infty}\int_0^{\pi/4} x^{\frac{2}{\nu}-1}\cos^{\frac{1}{\nu}-1}(u)\sin^{\frac{1}{\nu}-1}(u)(x\sin(u))^{\frac{1}{\nu}}(x^2\cos^2(u))e^{-\frac{x^2}{h}}dudx$$

$$= \frac{4\pi^2c^2}{\nu^2}(\int_0^{\infty} x^{\frac{3}{\nu}+1} e^{-\frac{x^2}{h}} dx)(\int_0^{\pi/4} \sin^{\frac{2}{\nu}-1}(u)\cos^{\frac{1}{\nu}+1}(u)du).$$

The first integral in the product above is

$$\int_0^{\infty} x^{\frac{3}{\nu}+1} e^{-\frac{x^2}{h}} dx = \frac{1}{2}\int_0^{\infty} y^{\frac{3}{2\nu}} e^{-\frac{y}{h}} dy$$

$$= \frac{1}{2}\Gamma(\frac{3}{2\nu}+1)\, h^{\frac{3}{2\nu}+1}.$$

The second integral is

$$\int_0^{\pi/4} \sin^{\frac{2}{\nu}-1}(u)\cos^{\frac{1}{\nu}+1}(u)du = \int_0^{1/2} \frac{(w^{1/2})^{\frac{2}{\nu}-1}((1-w)^{1/2})^{\frac{1}{\nu}+1}}{2w^{1/2}(1-w)^{1/2}}\,dw$$

$$= \frac{1}{2}\int_0^{1/2} w^{\frac{1}{\nu}-1}(1-w)^{\frac{1}{2\nu}}\,dw,$$

using the transformation $w = \sin^2(u)$. Hence,

$$E_{P_n}[G(Z_i)Z_i^{2\nu-1}] = \frac{4\pi^2c^2}{\nu^2}\left(\frac{1}{2}\Gamma\left(\frac{3}{2\nu}+1\right)h^{\frac{3}{2\nu}+1}\right)\left(\frac{1}{2}B_\nu\left(\frac{1}{2}\right)\right),$$

where (3.2.41)

$$B_\nu\left(\frac{1}{2}\right) = \int_0^{1/2} w^{\frac{1}{\nu}-1}(1-w)^{\frac{1}{2\nu}}\,dw.$$

It follows, using 3.2.39 - 3.2.41, that

$$\mathrm{Cov}_{P_n}[\bar{S}_i, L_i] = \frac{\nu K}{\sqrt{2}} \frac{\Gamma\left(\frac{3}{2\nu}+1\right)\left(\Gamma\left(\frac{2}{\nu}\right)\right)^{1/2}\beta_\nu\left(\frac{1}{2}\right)}{\left(\Gamma\left(\frac{1}{\nu}\right)\right)^{5/2}}$$

and (3.2.42)

$$\gamma_{12} = \mathrm{Cov}_{P_n}[\ell_1\sqrt{6}\,\bar{C}_i + \ell_2\sqrt{6}\,\bar{S}_i, L_i]$$

$$= \ell_2\sqrt{6}\,\mathrm{Cov}_{P_n}[\bar{S}_i, L_i].$$

We have found by Le Cam's lemma that $S_n = \ell_1(6n^{-1})^{1/2}C_{MB}+\ell_2(6n^{-1})^{1/2}S_{MB}$ is asymptotically normal with mean γ_{12} and variance $\ell_1^2+\ell_2^2$, under Q_n. Since $\mathrm{Cov}_{P_n}[\bar{C}_1, L_1] = 0$, it follows that $(6n^{-1})^{1/2}(C_{MB}, S_{MB})$ is under Q_n asymptotically bivariate normal with mean vector $(0, \sqrt{6}\ \mathrm{Cov}_{P_n}[\bar{S}_1, L_1])$ and covariance matrix I_2. We conclude that under Q_n

$$MB = (6n^{-1})^2(C_{MB}^2+S_{MB}^2)$$

is asymptotically Chi-square with two degrees of freedom and noncentrality parameter

$$\delta_{MB}^2(\nu) = (\sqrt{6}\ \mathrm{Cov}_{P_n}[\bar{S}_1, L_1])^2$$

$$= \left(\frac{\nu K\sqrt{3}\ \Gamma(\frac{3}{2\nu}+1)(\Gamma(\frac{2}{\nu}))^{\frac{1}{2}}\ \beta_\nu(\frac{1}{2})}{(\Gamma(\frac{1}{\nu}))^{5/2}} \right)^2 . \square \qquad (3.2.43)$$

3.3 Asymptotic Distribution of Blumen's Sign Test

Blumen (1958) found the asymptotic null distribution of his statistic, B, to be Chi-square with two degrees of freedom. We will derive this result in a different fashion so that we can again apply Le Cam's third lemma to find the asymptotic distribution of B under the contiguous alternatives considered in section 3.2. Using the notation in the previous section, recall that

$$B = 2n(C_B^2+S_B^2),$$

where (3.3.1)

$$C_B = \frac{1}{n} \sum_{i=1}^{n} a_i \cos(\pi H_n(\phi_i))$$

and

$$S_B = \frac{1}{n} \sum_{i=1}^{n} a_i \sin(\pi H_n(\phi_i)).$$

Let us define

$$\bar{C}_B = \frac{1}{n} \sum_{i=1}^{n} \bar{C}_{Bi} \quad \text{and} \quad \bar{S}_B = \frac{1}{n} \sum_{i=1}^{n} \bar{S}_{Bi},$$

where (3.3.2)

$$\bar{C}_{Bi} = a_i \cos(\pi H(\phi_i))$$

and

$$\bar{S}_{Bi} = a_i \sin(\pi H(\phi_i)).$$

In the following lemma we show that $\sqrt{n}(C_B, S_B)$ and $\sqrt{n}(\bar{C}_B, \bar{S}_B)$ have the same asymptotic null distribution.

Lemma 3.3.3. Suppose $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ are independently and identically distributed from a continuous bivariate distribution with median $\underset{\sim}{\theta} = (\theta_1, \theta_2)'$ satisfying 1.1.4. Under H_o: $\underset{\sim}{\theta} = \underset{\sim}{0}$,

$$\sqrt{n}(C_B-\bar{C}_B,\ S_B-\bar{S}_B) \overset{P}{\to} (0,\ 0),$$

as $n \to \infty$.

Proof. As in the proof of Lemma 3.1.15, write

$$E_{H_o}[\{\sqrt{n}(C_B-\bar{C}_B)\}^2]$$

$$= nE_{H_o}[\{\frac{1}{n}\sum_{i=1}^{n} a_i[\cos(\pi H_n(\phi_i))-\cos(\pi H(\phi_i))]\}^2]$$

$$= n^{-1}E_{H_o}[\sum_{i=1}^{n} a_i^2[\cos(\pi H_n(\phi_i))-\cos(\pi H(\phi_i))]^2]$$

$$+ n^{-1}E_{H_o}[\sum_{i\neq j} a_i a_j[\cos(\pi H_n(\phi_i))-\cos(\pi H(\phi_i))][\cos(\pi H_n(\phi_j))-\cos(\pi H(\phi_j))]. \tag{3.3.4}$$

Recalling that $a_1,\ldots,a_n$ are i.i.d. under H_o with $E_{H_o}[a_i] = 0$, and that $a_1,\ldots,a_n$ are independent of $\phi_1,\ldots,\phi_n$, it follows that the second term in the sum above is equal to zero. Hence,

$$E_{H_o}[\{\sqrt{n}(C_B-\bar{C}_B)\}^2] = E_{H_o}[\{\cos(\pi H_n(\phi_i))-\cos(\pi H(\phi_i))\}^2]. \tag{3.3.5}$$

By Glivenko-Cantelli, as in 3.1.20, $H_n(\phi_i)-H(\phi_i) \overset{H_o}{=} o_p(1)$, hence

$$\cos(\pi H_n(\phi_i))-\cos(\pi H(\phi_i)) \overset{H_o}{=} o_p(1).$$

Using Lebesgue Dominated Convergence Theorem it follows that the expression in 3.3.5 converges to zero, as $n \to \infty$. This shows that

$$\sqrt{n}(C_B - \bar{C}_B) \overset{H_o}{=} o_p(1),$$

and, in an analogous fashion, it is seen that

$$\sqrt{n}(S_B - \bar{S}_B) \overset{H_o}{=} o_p(1). \quad \square$$

Lemma 3.3.3 readily shows that the asymptotic distribution of B is Chi-square with two degrees of freedom. It is easily verified that under H_o $\mathrm{Cov}[\bar{C}_{Bi}, \bar{S}_{Bi}] = 0$, $E[\bar{C}_{Bi}] = E[\bar{S}_{Bi}] = 0$, and $\mathrm{Var}[\bar{C}_{Bi}] = \mathrm{Var}[\bar{S}_{Bi}] = \frac{1}{2}$. Hence by the Multivariate Central Limit Theorem the asymptotic null distribution of $\sqrt{2n}(\bar{C}_B, \bar{S}_B)$ is bivariate normal with mean $\underset{\sim}{0}$ and covariance matrix I_2. The asymptotic null distribution of B follows.

In the following theorem we use Lemma 3.3.3 to find the asymptotic distribution of B under the contiguous alternatives of section 3.2.

Theorem 3.3.6. Let $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ be independently and identically distributed from a density function given by 3.2.1, with $\Sigma = \mathrm{Var}[\underset{\sim}{Y}_i] = I_2$. If $\nu > 1/2$, then under H_n: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n = (0, Kn^{-1/2})'$, B is asymptotically Chi-square with two degrees of freedom and noncentrally parameter

$$\delta_B^2(\nu) = \left(\frac{\nu K \, \Gamma(\frac{1}{2\nu} + 1)(\Gamma(\frac{2}{\nu}))^{1/2}}{(\Gamma(\frac{1}{\nu}))^{3/2}} \right)^2 \qquad (3.3.7)$$

where

$$\Gamma(w) = \int_0^\infty u^{w-1} e^{-u} du, \qquad w > 0.$$

Proof. We will follow the procedure and use the same notation given in the proof of Theorem 3.2.29. Applying Le Cam's third lemma, we need to show that

$$(\ell_1\sqrt{2n}C_B+\ell_2\sqrt{2n}S_B,\ \log L_n)$$

is asymptotically bivariate normal under P_n. Using Lemmas 3.2.8 and 3.3.3, it suffices to show that

$$(\ell_1\sqrt{2n}\ \bar{C}_B+\ell_2\sqrt{2n}\ \bar{S}_B,\ L_n^*)$$

is asymptotically bivariate normal under P_n, where L_n^* is given by 3.2.24. This is immediate from the Multivariate Central Limit Theorem. Having already verified, in Lemma 3.2.23, that the parameters of the asymptotic P_n distribution of $\log L_n$ satisfy the relationship required in Le Cam's lemma, we conclude that $\ell_1\sqrt{2n}\ \bar{C}_B+\ell_2\sqrt{2n}\ \bar{S}_B$ and therefore, $\ell_1\sqrt{2n}\ C_B+\ell_2\sqrt{2n}\ S_B$, are each asymptotically normal under Q_n with mean

$$E_{P_n}[\ell_1\sqrt{2}\ \bar{C}_{Bi}+\ell_2\sqrt{2}\ \bar{S}_{Bi}] + \mathrm{Cov}_{P_n}[(\ell_1\sqrt{2}\ \bar{C}_{Bi}+\ell_2\sqrt{2}\ \bar{S}_{Bi}),L_i] \tag{3.3.8}$$

and variance

$$\mathrm{Var}_{P_n}[\ell_1\sqrt{2}\ \bar{C}_{Bi}+\ell_2\sqrt{2}\ \bar{S}_{Bi}],$$

where L_i is given by 3.2.24. It is easily verified that

$$E_{P_n}[\ell_1\sqrt{2}\ \bar{C}_{Bi}+\ell_2\sqrt{2}\ \bar{S}_{Bi}] = 0$$

and (3.3.9)

$$\mathrm{Var}_{P_n}[\ell_1\sqrt{2}\ \bar{C}_{Bi}+\ell_2\sqrt{2}\ \bar{S}_{Bi}] = \ell_1^2+\ell_2^2.$$

We find

$$\begin{aligned}\mathrm{Cov}_{P_n}[\bar{C}_{Bi},\ L_i] &= E_{P_n}[\bar{C}_{Bi}L_i]\\ &= 2\nu K h^{-1}E_{P_n}[a_i\cos(\pi H(\phi_i))\sin(\phi_i')]E_{P_n}[Z_i^{2\nu-1}]\\ &= 0,\end{aligned} \tag{3.3.10}$$

using 3.2.35 and 3.2.37. In addition,

$$\begin{aligned}\mathrm{Cov}_{P_n}[\bar{S}_{Bi},\ L_i] &= E_{P_n}[\bar{S}_{Bi}L_i]\\ &= 2\nu K h^{-1}E_{P_n}[a_i\sin(\pi H(\phi_i))\sin(\phi_i')]E_{P_n}[Z_i^{2\nu-1}]\\ &= 2\nu K h^{-1}(1/2)\left(\frac{\pi c}{\nu}\,\Gamma\left(\frac{1}{2\nu}+1\right)h^{\left(\frac{1}{2\nu}+1\right)}\right),\end{aligned} \tag{3.3.11}$$

using 3.2.21 and 3.2.40. Thus, using 3.3.8 - 3.3.10, $\ell_1\sqrt{2n}\ C_B+\ell_2\sqrt{2n}\ S_B$ is asymptotically normal under Q_n with mean $\ell_2\sqrt{2}\mathrm{Cov}_{P_n}[\bar{S}_{Bi},\ L_i]$ and variance $\ell_1^2+\ell_2^2$. Therefore, under Q_n, $\sqrt{2n}(C_B,\ S_B)$ is asymptotically

bivariate normal with mean $(0, \sqrt{2}\ \mathrm{Cov}[\bar{S}_{Bi}, L_i])$ and covariance matrix, I_2. The asymptotic distribution of $B = 2n(C_B^2+S_B^2)$ under Q_n is then noncentral Chi-square with two degrees of freedom and noncentrally parameter,

$$\delta_B^2(\nu) = 2(\mathrm{Cov}[\bar{S}_{Bi}, L_i])^2$$

$$= 2\left(2\nu K h^{-1}(\tfrac{1}{2})(\tfrac{\pi c}{\nu}\,\Gamma(\tfrac{1}{2\nu}+1)\,h^{\frac{1}{2\nu}+1})\right)^2$$

$$= \left(\frac{\nu K\,\Gamma(\frac{1}{2\nu}+1)\,(\Gamma(\frac{2}{\nu}))^{1/2}}{(\Gamma(\frac{1}{\nu}))^{3/2}}\right)^2. \qquad (3.3.12)$$

CHAPTER 4
COMPARISONS OF ONE-SAMPLE TEST STATISTICS

4.1 Asymptotic Relative Efficiencies

In this section we use Pitman asymptotic relative efficiencies to make comparisons among Hotelling's T^2, the sign statistics of Blumen and Hodges, and the proposed modification of Blumen's statistic. (See Chapter 1 for definitions of Hotelling's T^2 and Hodges' statistic, H.) Because these statistics are all affine-invariant we may, without loss of generality, make the simplifying assumption that the population covariance matrix is the identity. Further, alternatives of the form $\underset{\sim}{\theta} = (K_1, K_2)'$ can be transformed to equivalent alternatives of the form $\underset{\sim}{\theta} = (0, (K_1^2+K_2^2)^{1/2})$ by a careful choice of an orthogonal transformation (rotation) of the observations. Hence we will, without loss of generality, consider alternatives of the form $\underset{\sim}{\theta} = (0, \Delta)'$ and we will apply the asymptotic results under the contiguous alternatives discussed in Chapter 3.

The Pitman approach to asymptotic relative efficiency compares two test sequences S_{K_n} and $T_{K_n^*}$ as the alternative hypothesis, H_n: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n$, approaches the null, which we are taking to be H_o: $\underset{\sim}{\theta} = \underset{\sim}{0}$. The subscripts K_n and K_n^* are the sample sizes for tests S_{K_n} and $T_{K_n^*}$, respectively. Let Power $S_{K_n}(\underset{\sim}{\theta}_n)$ and Power $T_{K_n^*}(\underset{\sim}{\theta}_n)$ denote the probabilities that tests S_{K_n} and $T_{K_n^*}$, respectively, reject

H_o when $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n$. Assume that K_n and K_n^* are such that the limiting significance levels are the same for each test sequence, as are the limiting powers:

$$\lim_{n\to\infty} \text{Power } S_{K_n}(\underset{\sim}{\theta}_n) = \lim_{n\to\infty} \text{Power } T_{K_n^*}(\underset{\sim}{\theta}_n) = \beta,$$

where $0 < \alpha < \beta < 1$, and α denotes the limit of the significance levels of S_{K_n} and $T_{K_n^*}$, as $n \to \infty$. The Pitman asymptotic relative efficiency of S_{K_n} relative to $T_{K_n^*}$ is

$$\text{ARE}(S, T) = \lim_{n\to\infty} \frac{K_n^*}{K_n}, \tag{4.1.1}$$

provided the limit exists and is the same for all $\{\underset{\sim}{\theta}_n\}$, $\{K_n\}$ and $\{K_n^*\}$ satisfying the conditions above. (See, e.g., Randles and Wolfe, 1979, p. 144.)

Hannan (1956) shows that if, under H_n, the test sequences S_{K_n} and $T_{K_n^*}$ are asymptotically noncentral Chi-square with ν degrees of freedom and noncentrality parameters, δ_S^2 and δ_T^2, respectively, then

$$\text{ARE}(S, T) = \frac{\delta_S^2}{\delta_T^2}. \tag{4.1.2}$$

It is well-known that the asymptotic distribution of Hotelling's T^2, under H_n: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n = (0, Kn^{-1/2})$, is noncentral Chi-square with two degrees of freedom and noncentrality parameter,

$$\delta_T^2 = K^2. \tag{4.1.3}$$

(See, e.g., Puri and Sen, 1971, p. 173.) We can use, then,

Theorems 3.2.29 and 3.3.6 to find the asymptotic relative efficiencies of B and MB relative to T^2. These results are stated in the following theorem.

Theorem 4.1.4. Suppose $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independently and identically distributed from a density given by 3.2.1, with $\Sigma = I_2$. For testing

$$H_o: \underset{\sim}{\theta} = \underset{\sim}{0} \quad \text{against} \quad H_a: \underset{\sim}{\theta} = \underset{\sim}{\theta}_n,$$

if $\nu > 1/2$, then

$$ARE(B, T^2) = \frac{\delta_B^2(\nu)}{\delta_T^2} = \left(\frac{\nu\Gamma(\frac{1}{2\nu}+1)(\Gamma(\frac{2}{\nu}))^{1/2}}{(\Gamma(\frac{1}{\nu}))^{3/2}} \right)^2$$

and

$$ARE(MB, T^2) = \frac{\delta_{MB}^2(\nu)}{\delta_T^2} = \left(\frac{\nu\sqrt{3}\ \Gamma(\frac{3}{2\nu}+1)(\Gamma(\frac{2}{\nu}))^{1/2}\beta_\nu(\frac{1}{2})}{(\Gamma(\frac{1}{\nu}))^{5/2}} \right)^2$$

where

$$\Gamma(w) = \int_0^\infty u^{w-1}e^{-u}du, \ w > 0,$$

and

$$\beta_\nu(\tfrac{1}{2}) = \int_0^{\frac{1}{2}} u^{\frac{1}{\nu}-1}(1-u)^{\frac{1}{2\nu}}du.$$

Table 4.1 provides the Pitman asymptotic relative efficiencies of B and MB relative to T^2, for selected values of ν.

Table 4.1
Pitman Asymptotic Relative Efficiencies

ν	ARE(B, T^2)	ARE(MB, T^2)
10.0	.50530	1.33231
5.0	.51872	1.22101
1.0	.78540	.98464
.8	.91839	.99574
.7	1.02988	1.01488
$.5^+$	1.50000	1.12500

We see that although Hotelling's T^2 performs best for normal alternatives ($\nu = 1.0$), MB appears to be quite competitive. Except for distributions very close to normal, MB is asymptotically more powerful than Hotelling's T^2. While Blumen's statistic, B, performs poorly for lighter-tailed distributions, it performs better than Hotelling's T^2 and MB for heavy-tailed distributions. The Pitman asymptotic relative efficiency has not been found for Hodges' statistic. However, Bahadur (1960) notes that under certain conditions the limiting Bahadur efficiency coincides with that of Pitman. Killeen and Hettmansperger (1972) found the limiting Bahadur efficiency of Hodges' statistic, H, relative to Hotelling's T^2 to be $\frac{2}{\pi}$ under normal alternatives. It appears then that under normal alternatives B, MB, and T^2 are each superior to H asymptotically.

We have also displayed in Table 4.2 conjectured Pitman asymptotic relative efficiencies when $0 < \nu \leq \frac{1}{2}$, using the form of the efficiencies given in Theorem 4.1.4. For heavy-tailed distributions, further evidence of the superior performance of B and the inferior performance of T^2 is indicated.

Table 4.2
Conjectured Pitman Asymptotic
Relative Efficiencies

ν	ARE(B, T^2)	ARE(MB, T^2)
.4	2.09838	1.27541
.3	3.69648	1.64385
.2	11.59689	3.00306
.1	366.57838	25.03483

4.2 Monte Carlo Results

Additional comparisons of several one-sample test procedures will be made in this section using Monte Carlo methods.[1] The affine-invariant statistics of the last section, namely B, H, T^2, and MB are examined along with two statistics which are not affine-invariant. Bennett's sign test statistic B1 (see 1.1.5) is included along with his signed-rank test, which was also introduced in Chapter 1. The

[1] All calculations and random variable generations were performed with an IBM-4341 running on a vm/cms operating system. Fortran 77 and several IMSL library subroutines were used.

precise form of the signed-rank test is

$$B2 = [(\frac{1}{n+1})(U^+, V^+)]'(nS)^{-1}[(\frac{1}{n+1})(U^+, V^+)],$$

where U^+ and V^+ are the signed rank statistics for the marginal observations as described in 1.1.6, and S is a consistent estimator of the asymptotic covariance matrix of $\frac{1}{\sqrt{n}(n+1)}(U^+, V^+)$. The estimator, S, we use has diagonal elements equal to $\frac{2n+1}{6(n+1)}$, and off diagonal elements equal to

$$\frac{1}{n(n+1)^2}\sum_{i=1}^{n} \text{Rank}(|Z_{1i}|)\text{Rank}(|Z_{2i}|)I[Z_{1i} > 0]I[Z_{2i} > 0].$$

(See, e.g., Hettmansperger, 1984, p. 284.) These six test statistics were compared for samples taken from several distributions, most of which were of the form given in 3.2.1. Additional comparisons were made using samples from Pearson Type II and Type VII (see Chapter 6 of Johnson, 1987), and mixtures of bivariate normal distributions. (A bivariate normal mixture is obtained by selecting at random one of two bivariate normal distributions. Each of the sample elements is selected with probability p from the first distribution and with probability 1-p from the second distribution. See Johnson, 1987, Chapter 4.) In each Monte Carlo simulation, the proportion of times each test statistic exceeded the upper α-percentile of its asymptotic null distribution is reported. The rejection proportions for T^2 using critical values from its distribution under normality (namely, a

multiple of $F_{2,n-2}$) are given in parentheses. The alternatives considered were of the form $\theta_{\underset{\sim}{a}} = (\theta_a, \theta_{\underset{\sim}{a}})'$, which results in no loss in generality for the affine-invariant statistics. For most of the simulations, values for θ_a were selected so that $\theta_{\underset{\sim}{a}}'\Sigma^{-1}\theta_{\underset{\sim}{a}}$ equaled $.2^2$, $.4^2$, or $.6^2$, where $\Sigma = \mathrm{Var}[\underset{\sim}{Y}]$ was taken to be the identity. Heavy-tailed distributions were treated somewhat differently. For settings in which $\mathrm{Var}[\underset{\sim}{Y}]$ does not exist, the random variables were scaled to produce readily comparable results. For the heaviest-tailed distribution examined from the class given in 3.2.1 ($\nu = .10$), the amount of translation was altered. Significance levels were chosen to agree with available asymptotic percentiles for Hodges' null distribution. (See Joffe and Klotz, 1962.)

In Table 4.3, the results from the first Monte Carlo study are presented. As indicated, 1000 samples of size 25 were obtained from five members of the class of distributions given by 3.2.1, with $\Sigma = I_2$. Heavy-tailed distributions, $\nu = .10$ and .50, the bivariate normal distribution, $\nu = 1.00$, and light-tailed distributions, $\nu = 50.00$ and 120.00, were included. The bivariate medians of the underlying distributions and the nominal significance level, α, are indicated. Although it is difficult to compare empirical powers for statistics with varying rejection proportions under the null hypothesis, it appears that the Monte Carlo results tend to agree with the asymptotic results of the previous section. Hotelling's T^2 has the greatest power for normal alternatives. For distributions close to normal, T^2 and MB perform similarly, both performing better than

Table 4.3
One-Sample Monte Carlo Results with n = 25 and α = .05318

Bivariate medians	Statistics					
	B	H	T^2	MB	B1	B2
			ν = .10			
(.000,.000)	.052	.045	.047 (.018)	.060	.047	.051
(.014,.014)	.144	.136	.061 (.026)	.087	.128	.086
(.028,.028)	.304	.270	.074 (.041)	.138	.278	.148
(.042,.042)	.478	.437	.096 (.064)	.211	.419	.246
			ν = .50			
(.000,.000)	.052	.045	.080 (.049)	.060	.047	.048
(.141,.141)	.154	.140	.192 (.142)	.151	.118	.160
(.283,.283)	.485	.449	.515 (.459)	.443	.398	.453
(.424,.424)	.809	.760	.806 (.745)	.726	.726	.776
			ν = 1.00			
(.000,.000)	.052	.045	.084 (.054)	.058	.047	.048
(.141,.141)	.107	.105	.181 (.134)	.135	.091	.119
(.283,.283)	.313	.289	.468 (.381)	.378	.268	.361
(.424,.424)	.612	.574	.798 (.724)	.708	.528	.704
			ν = 50.00			
(.000,.000)	.052	.045	.080 (.062)	.060	.047	.051
(.141,.141)	.082	.076	.171 (.127)	.145	.069	.108
(.283,.283)	.202	.198	.448 (.362)	.441	.157	.306
(.424,.424)	.442	.401	.794 (.721)	.796	.366	.639
			ν = 120.00			
(.000,.000)	.052	.045	.088 (.054)	.061	.047	.048
(.141,.141)	.076	.072	.170 (.122)	.166	.057	.104
(.283,.283)	.140	.132	.431 (.351)	.514	.110	.302
(.424,.424)	.298	.265	.785 (.703)	.864	.234	.624

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample size: 25.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Blumen's statistic. For the heaviest-tailed distribution examined, Blumen's test appears to perform best and MB performs considerably better than T^2. The superiority of MB for very light-tailed distributions is shown. Hodges' test statistic, H, follows the same general pattern as Blumen's statistic, B, performing well for heavy-tailed distributions, but not well for light-tailed distributions. Bennett's sign test and signed-rank test perform well except for light- and heavy-tailed distributions, respectively. However, an examination of Table 4.4 indicates that the performances of Bennett's nonaffine-invariant tests depend highly on the covariance structure of the distribution. As the correlation decreases (from 0.00 to -.99), the empirical powers tend to decrease, becoming rather low in the most extreme case in which the correlation is -.99. (The direction that $\underset{\sim}{\theta}_a$ takes from the origin also plays a role in the performances of the statistics that are not affine-invariant.) Hence, we focus on B, H, T^2, and MB. In Tables 4.5 and 4.6 with significance levels of .08865 and .00881, respectively, the same general results are obtained.

Likewise, similar results are seen using samples of sizes 20, 15, 12, 10, and 8. (See Tables 4.7 through 4.11.) Since the asymptotic null distributions for Hodges' statistic are not suited for moderately sized samples, we have used the exact null distributions here. Hodges (1955) gives the null distribution of H for small samples. Klotz (1964) gives percentiles of the null distribution of Blumen's and Hodges' statistics for samples of sizes 8 through 12. We have included the available small sample critical values for Blumen's

Table 4.4
One-Sample Monte Carlo Results with
$n = 25$, $\alpha = .05318$, and Varying Correlation Structures

Bivariate medians	Statistics						
	B	H	T^2		MB	B1	B2
	Bivariate normal (unit variances, correlation = 0.0)						
(.000,.000)	.052	.045	.084	(.054)	.058	.047	.048
(.141,.141)	.107	.105	.181	(.134)	.135	.091	.119
(.283,.283)	.313	.289	.468	(.381)	.378	.268	.361
(.424,.424)	.612	.574	.798	(.724)	.708	.528	.704
	Bivariate normal (unit variances, correlation = -0.5)						
(.000,.000)	.051	.050	.077	(.052)	.054	.039	.054
(.100,.100)	.116	.100	.177	(.136)	.140	.087	.114
(.200,.200)	.277	.261	.452	(.368)	.364	.184	.311
(.300,.300)	.604	.551	.796	(.722)	.716	.386	.659
	Bivariate normal (unit variances, correlation = -0.99)						
(.000,.000)	.047	.047	.075	(.048)	.056	.019	.054
(.014,.014)	.107	.094	.167	(.121)	.128	.022	.088
(.028,.028)	.301	.270	.468	(.390)	.388	.022	.244
(.042,.042)	.615	.571	.774	(.724)	.707	.028	.470

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample size: 25.
Distributions: Bivariate normal.

Table 4.5
One-Sample Monte Carlo Results with n = 25 and α = .08865

Bivariate medians	Statistics					
	B	H	T^2	MB	B1	B2
			ν = .10			
(.000,.000)	.087	.050	.076 (.041)	.088	.113	.088
(.014,.014)	.203	.132	.085 (.047)	.137	.219	.134
(.028,.028)	.368	.248	.097 (.063)	.190	.359	.209
(.042,.042)	.525	.395	.126 (.085)	.266	.506	.306
			ν = .50			
(.000,.000)	.087	.050	.135 (.091)	.114	.113	.090
(.141,.141)	.224	.151	.242 (.202)	.224	.234	.219
(.283,.283)	.575	.414	.580 (.524)	.534	.533	.562
(.424,.424)	.844	.747	.840 (.799)	.790	.802	.834
			ν = 1.00			
(.000,.000)	.087	.050	.120 (.085)	.095	.113	.086
(.141,.141)	.170	.106	.234 (.188)	.204	.173	.179
(.283,.283)	.379	.276	.543 (.478)	.482	.368	.449
(.424,.424)	.688	.556	.838 (.793)	.784	.614	.763
			ν = 50.00			
(.000,.000)	.087	.050	.119 (.088)	.093	.113	.088
(.141,.141)	.141	.076	.244 (.197)	.226	.146	.168
(.283,.283)	.269	.186	.510 (.450)	.539	.253	.405
(.424,.424)	.541	.380	.833 (.789)	.854	.471	.722
			ν = 120.00			
(.000,.000)	.087	.050	.119 (.086)	.092	.113	.081
(.141,.141)	.122	.067	.227 (.177)	.255	.140	.171
(.283,.283)	.211	.132	.494 (.423)	.598	.215	.391
(.424,.424)	.387	.263	.812 (.751)	.902	.338	.679

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .08865.
Number of samples: 1000 from each distribution.
Sample size: 25.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Table 4.6
One-Sample Monte Carlo Results with n = 25 and α = .00881

Bivariate medians	Statistics						
	B	H	T^2		MB	B1	B2
			ν = .10				
(.000,.000)	.009	.002	.008	(.004)	.009	.004	.004
(.014,.014)	.030	.014	.011	(.005)	.014	.025	.015
(.028,.028)	.072	.045	.018	(.007)	.023	.069	.029
(.042,.042)	.172	.092	.020	(.012)	.039	.141	.063
			ν = .50				
(.000,.000)	.009	.002	.017	(.005)	.008	.004	.005
(.141,.141)	.041	.023	.071	(.040)	.038	.030	.041
(.283,.283)	.168	.117	.258	(.164)	.129	.151	.145
(.424,.424)	.440	.298	.586	(.449)	.338	.365	.428
			ν = 1.00				
(.000,.000)	.009	.002	.025	(.006)	.004	.004	.002
(.141,.141)	.029	.006	.077	(.042)	.024	.016	.023
(.283,.283)	.094	.043	.247	(.162)	.131	.072	.113
(.424,.424)	.268	.182	.543	(.401)	.325	.216	.330
			ν = 50.00				
(.000,.000)	.009	.002	.026	(.012)	.010	.004	.005
(.141,.141)	.018	.006	.066	(.040)	.028	.012	.025
(.283,.283)	.051	.029	.224	(.131)	.129	.042	.078
(.424,.424)	.158	.092	.520	(.379)	.402	.112	.262
			ν = 120.00				
(.000,.000)	.009	.002	.030	(.010)	.007	.004	.003
(.141,.141)	.016	.004	.062	(.037)	.032	.011	.019
(.283,.283)	.034	.020	.196	(.112)	.156	.028	.076
(.424,.424)	.088	.046	.462	(.339)	.501	.070	.207

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .00881.
Number of samples: 1000 from each distribution.
Sample size: 25.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Table 4.7
One-Sample Monte Carlo Results with n = 20 and α = .03044

Bivariate medians	Statistics					
	B	H	T^2		MB	B1	B2
			ν = .10				
(.000,.000)	.023	.032	.028	(.007)	.030	.032	.025
(.014,.014)	.073	.088	.034	(.009)	.044	.086	.038
(.028,.028)	.169	.184	.043	(.018)	.080	.180	.084
(.042,.042)	.282	.287	.072	(.033)	.118	.285	.152
			ν = .50				
(.000,.000)	.023	.032	.057	(.023)	.040	.032	.023
(.141,.141)	.086	.090	.141	(.082)	.096	.077	.077
(.283,.283)	.281	.265	.359	(.255)	.235	.247	.259
(.424,.424)	.543	.545	.627	(.524)	.491	.503	.530
			ν = 1.00				
(.000,.000)	.023	.032	.066	(.034)	.030	.032	.023
(.141,.141)	.066	.065	.134	(.068)	.073	.056	.053
(.283,.283)	.174	.174	.332	(.232)	.207	.162	.188
(.424,.424)	.391	.379	.622	(.508)	.463	.342	.458
			ν = 50.00				
(.000,.000)	.023	.032	.060	(.032)	.026	.032	.023
(.141,.141)	.045	.053	.115	(.069)	.071	.043	.045
(.283,.283)	.114	.118	.311	(.218)	.245	.110	.159
(.424,.424)	.259	.247	.608	(.474)	.525	.235	.378
			ν = 120.00				
(.000,.000)	.023	.032	.066	(.034)	.033	.032	.023
(.141,.141)	.043	.050	.127	(.077)	.105	.045	.055
(.283,.283)	.089	.088	.301	(.191)	.296	.080	.152
(.424,.424)	.175	.167	.569	(.430)	.597	.162	.340

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Hodges' rejection proportions were obtained using the exact small sample null distribution of H. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .03044.
Number of samples: 1000 from each distribution.
Sample size: 20.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Table 4.8
One-Sample Monte Carlo Results with n = 15 and α = .07050

Bivariate medians	Statistics					
	B	H	T^2	MB	B1	B2
			ν = .10			
(.000,.000)	.065	.064	.065 (.023)	.071	.073	.046
(.014,.014)	.122	.113	.070 (.028)	.093	.140	.071
(.028,.028)	.211	.207	.097 (.040)	.127	.237	.135
(.042,.042)	.302	.299	.133 (.063)	.173	.345	.191
			ν = .50			
(.000,.000)	.065	.064	.117 (.058)	.085	.073	.059
(.141,.141)	.129	.130	.197 (.116)	.132	.149	.112
(.283,.283)	.324	.320	.416 (.294)	.295	.305	.292
(.424,.424)	.592	.548	.664 (.540)	.528	.562	.556
			ν = 1.00			
(.000,.000)	.065	.064	.107 (.057)	.069	.073	.057
(.141,.141)	.090	.093	.164 (.102)	.106	.119	.094
(.283,.283)	.205	.213	.363 (.255)	.257	.230	.233
(.424,.424)	.391	.388	.628 (.503)	.493	.402	.460
			ν = 50.00			
(.000,.000)	.065	.064	.116 (.063)	.086	.073	.057
(.141,.141)	.075	.080	.174 (.110)	.134	.105	.092
(.283,.283)	.164	.155	.355 (.249)	.290	.163	.201
(.424,.424)	.303	.293	.607 (.482)	.557	.292	.421
			ν = 120.00			
(.000,.000)	.065	.064	.120 (.065)	.095	.073	.053
(.141,.141)	.078	.073	.176 (.106)	.153	.094	.092
(.283,.283)	.124	.129	.343 (.243)	.336	.137	.206
(.424,.424)	.222	.222	.588 (.446)	.599	.219	.384

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Hodges' rejection proportions were obtained using the exact small sample null distribution of H. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .07050.
Number of samples: 1000 from each distribution.
Sample size: 15.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Table 4.9
One-Sample Monte Carlo Results with n = 12 and α = .05859

Bivariate medians	Statistics							
	B		H	T^2		MB	B1	B2
				ν = .10				
(.000,.000)	.050	(.052)	.058	.062	(.020)	.059	.049	.045
(.014,.014)	.086	(.105)	.092	.073	(.020)	.078	.084	.054
(.028,.028)	.132	(.145)	.147	.095	(.040)	.116	.149	.091
(.042,.042)	.214	(.230)	.214	.127	(.058)	.144	.217	.134
				ν = .50				
(.000,.000)	.050	(.052)	.058	.119	(.055)	.069	.049	.045
(.141,.141)	.090	(.103)	.104	.187	(.102)	.115	.079	.091
(.283,.283)	.232	(.248)	.249	.366	(.219)	.248	.202	.207
(.424,.424)	.423	(.447)	.440	.590	(.431)	.402	.388	.406
				ν = 1.00				
(.000,.000)	.050	(.057)	.058	.123	(.059)	.065	.049	.050
(.141,.141)	.078	(.086)	.084	.179	(.088)	.097	.081	.074
(.283,.283)	.152	(.161)	.156	.321	(.195)	.206	.144	.160
(.424,.424)	.295	(.314)	.319	.542	(.369)	.359	.231	.311
				ν = 50.00				
(.000,.000)	.050	(.056)	.058	.133	(.069)	.073	.049	.054
(.141,.141)	.062	(.072)	.075	.182	(.092)	.105	.063	.064
(.283,.283)	.120	(.126)	.127	.318	(.186)	.210	.094	.140
(.424,.424)	.225	(.241)	.242	.530	(.361)	.383	.167	.270
				ν = 120.00				
(.000,.000)	.050	(.058)	.058	.144	(.068)	.066	.049	.046
(.141,.141)	.055	(.059)	.065	.157	(.092)	.100	.059	.072
(.283,.283)	.090	(.102)	.099	.295	(.164)	.223	.079	.119
(.424,.424)	.147	(.159)	.167	.498	(.311)	.425	.126	.237

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Hodges' rejection proportions were obtained using the exact small sample null distribution of H. Values in parentheses are the rejection proportions for Hotelling's T^2 and Blumen's test using the $F_{2,n-2}$ distribution and the exact small sample null distribution for B, respectively. Blumen's test was randomized when using the small sample distribution.

Nominal significance level: .05859.
Number of samples: 1000 from each distribution.
Sample size: 12.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Table 4.10
One-Sample Monte Carlo Results with n = 10 and α = .03906

Bivariate medians	Statistics							
	B		H	T^2		MB	B1	B2
				ν = .10				
(.000,.000)	.039	(.039)	.045	.045	(.007)	.039	.051	.013
(.014,.014)	.067	(.067)	.073	.058	(.011)	.061	.076	.024
(.028,.028)	.106	(.106)	.104	.075	(.019)	.072	.121	.043
(.042,.042)	.164	(.164)	.151	.110	(.033)	.109	.163	.072
				ν = .50				
(.000,.000)	.039	(.039)	.041	.111	(.027)	.050	.051	.017
(.141,.141)	.068	(.068)	.068	.158	(.064)	.071	.083	.043
(.283,.283)	.155	(.155)	.139	.301	(.153)	.150	.161	.108
(.424,.424)	.294	(.294)	.257	.500	(.274)	.241	.283	.229
				ν = 1.00				
(.000,.000)	.039	(.039)	.046	.118	(.048)	.052	.051	.028
(.141,.141)	.064	(.064)	.053	.149	(.059)	.071	.061	.040
(.283,.283)	.112	(.112)	.108	.272	(.121)	.115	.110	.077
(.424,.424)	.197	(.197)	.162	.455	(.242)	.197	.187	.149
				ν = 50.00				
(.000,.000)	.039	(.039)	.042	.129	(.052)	.050	.051	.026
(.141,.141)	.052	(.052)	.055	.147	(.067)	.062	.058	.038
(.283,.283)	.087	(.087)	.079	.246	(.111)	.106	.086	.063
(.424,.424)	.133	(.133)	.121	.410	(.202)	.192	.142	.118
				ν = 120.00				
(.000,.000)	.039	(.039)	.047	.126	(.050)	.053	.051	.025
(.141,.141)	.049	(.049)	.053	.150	(.066)	.079	.053	.038
(.283,.283)	.072	(.072)	.059	.231	(.117)	.132	.071	.060
(.424,.424)	.104	(.104)	.090	.385	(.188)	.232	.108	.109

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Hodges' results based on a randomized test using the exact small sample null distribution. Values in parentheses are the rejection proportions for Hotelling's T^2 and Blumen's test using the $F_{2,n-2}$ distribution and the exact small sample null distribution for B, respectively.
Nominal significance level: .03906.
Number of samples: 1000 from each distribution.
Sample size: 10.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

Table 4.11
One-Sample Monte Carlo Results with n = 8 and α = .06250

Bivariate medians	Statistics							
	B		H	T^2		MB	B1	B2
				ν = .10				
(.000,.000)	.068	(.068)	.068	.108	(.024)	.090	.017	.032
(.014,.014)	.086	(.086)	.086	.118	(.025)	.098	.028	.045
(.028,.028)	.133	(.133)	.133	.138	(.034)	.136	.053	.076
(.042,.042)	.167	(.167)	.167	.160	(.057)	.152	.085	.114
				ν = .50				
(.000,.000)	.068	(.068)	.068	.167	(.051)	.089	.017	.042
(.141,.141)	.095	(.095)	.095	.226	(.066)	.114	.032	.064
(.283,.283)	.181	(.181)	.181	.366	(.153)	.179	.068	.135
(.424,.424)	.306	(.306)	.306	.525	(.294)	.262	.156	.259
				ν = 1.00				
(.000,.000)	.068	(.068)	.068	.180	(.075)	.088	.017	.050
(.141,.141)	.084	(.084)	.084	.218	(.081)	.104	.021	.058
(.283,.283)	.137	(.137)	.137	.317	(.136)	.171	.044	.095
(.424,.424)	.231	(.231)	.231	.482	(.243)	.251	.089	.191
				ν = 50.00				
(.000,.000)	.068	(.068)	.068	.183	(.079)	.092	.017	.045
(.141,.141)	.079	(.079)	.079	.226	(.091)	.110	.019	.060
(.283,.283)	.115	(.115)	.115	.313	(.138)	.158	.036	.095
(.424,.424)	.187	(.187)	.187	.440	(.219)	.235	.080	.152
				ν = 120.00				
(.000,.000)	.068	(.068)	.068	.178	(.076)	.092	.017	.055
(.141,.141)	.072	(.072)	.072	.218	(.094)	.114	.018	.064
(.283,.283)	.093	(.093)	.093	.313	(.134)	.166	.024	.082
(.424,.424)	.142	(.142)	.142	.407	(.214)	.263	.046	.128

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Hodges' rejection proportions were obtained using the exact small sample null distribution of H. Values in parentheses are the rejection proportions for Hotelling's T^2 and Blumen's test using the $F_{2,n-2}$ distribution and the exact small sample null distribution for B, respectively.
Nominal significance level: .06250.
Number of samples: 1000 from each distribution.
Sample size: 8.
Distributions: Class given in 3.2.1, $\Sigma = I_2$.

statistic in our Monte Carlo study for samples of size 8, 10, and 12. The significance levels selected were chosen from the natural levels of Blumen's and/or Hodges' null distribution. In some cases, either Blumen's test or Hodges' test was randomized so that the significance levels would agree. (There are relatively few natural significance levels for Hodges' statistic for small samples.) An examination of the empirical significance levels obtained suggests that samples of size 10 or larger are needed to effectively use the asymptotic null distribution of MB.

In Tables 4.12 we present the empirical powers for samples from light-tailed Pearson Type II distributions and heavy-tailed Pearson Type VII distributions. Again, the same general trends are indicated. Finally in Table 4.13 we examine samples from bivariate normal mixtures violating the assumption of elliptical symmetry required for the asymptotic results of Chapter 3. Nevertheless, MB appears to be rather robust. The contours of the first two mixtures are illustrated on page 56 of Johnson (1987). (The parameters of the distributions and the mixing probabilities are indicated in Table 4.13 for each mixture. The means have been translated from that shown by Johnson so that the bivariate null mean of the mixture is $\underset{\sim}{0}$.) For these asymmetric distributions, MB performs similarly to Hotelling's T^2 and better than the other competitors. The bimodal distribution used in Mixture 3 (Johnson, 1987, p. 61) results in a relatively large significance level for MB, but significantly greater power compared to the power of its competitors. The last two mixtures

Table 4.12
One-Sample Monte Carlo Results for Pearson Type II and Type VII Distributions with n = 25 and α = .05318

Bivariate medians	Statistics					
	B	H	T^2		MB	B1	B2
			Bivariate Normal				
(.000,.000)	.056	.043	.088	(.062)	.067	.056	.051
(.141,.141)	.119	.108	.187	(.128)	.135	.103	.119
(.283,.283)	.326	.295	.478	(.395)	.394	.248	.363
(.424,.424)	.600	.559	.794	(.727)	.702	.508	.688
			Pearson Type II (m = 1)				
(.000,.000)	.056	.043	.096	(.065)	.068	.056	.049
(.141,.141)	.092	.080	.181	(.131)	.153	.084	.109
(.283,.283)	.221	.205	.463	(.387)	.445	.186	.327
(.424,.424)	.441	.405	.778	(.706)	.787	.344	.632
			Pearson Type II (m = .5)				
(.000,.000)	.056	.043	.099	(.063)	.068	.056	.051
(.141,.141)	.085	.075	.173	(.128)	.197	.073	.112
(.283,.283)	.195	.176	.458	(.377)	.545	.163	.325
(.424,.424)	.379	.353	.780	(.710)	.840	.294	.641
			Pearson Type VII (m = 2)				
(.000,.000)	.056	.043	.068	(.047)	.068	.056	.055
(.141,.141)	.167	.154	.160	(.112)	.155	.138	.146
(.283,.283)	.476	.444	.388	(.325)	.411	.401	.439
(.424,.424)	.791	.750	.624	(.561)	.666	.713	.741
			Pearson Type VII (m = .5, Cauchy)				
(.000,.000)	.056	.043	.046	(.025)	.071	.059	.056
(.141,.141)	.294	.274	.113	(.069)	.212	.184	.238
(.283,.283)	.712	.669	.273	(.218)	.491	.463	.650
(.424,.424)	.933	.910	.443	(.382)	.708	.671	.899

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample size: 25.

Table 4.12--continued.

Distributions:

Bivariate Pearson Type II

$$\frac{\Gamma(m+2)}{\Gamma(m+1)\pi} |\Sigma|^{-1/2}\{1-(\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})\}^{m} \text{ if } (\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta}) \leq 1,\ m > -1.$$

Bivariate Pearson Type VII

$$\frac{\Gamma(m)}{\Gamma(m-1)\pi} |\Sigma|^{-1/2}\{1+(\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})\}^{-m},\ m > 1,\ \underset{\sim}{y} \in R^{2}.$$

Table 4.13
One-Sample Monte Carlo Results for Bivariate Normal Mixtures
with n = 25 and α = .05318

Bivariate medians	Statistics						
	B	H	T^2		MB	B1	B2
	Mixture 1 ($p = .5$, $\mu_{11} = \mu_{12} = -1.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = 1.0$, $\sigma_{21} = \sigma_{22} = 1.0$, $\rho_2 = 0.0$)						
(.000,.000)	.054	.051	.082	(.060)	.054	.050	.051
(.141,.141)	.060	.061	.115	(.078)	.088	.057	.071
(.283,.283)	.099	.090	.212	(.152)	.165	.093	.133
(.424,.424)	.170	.149	.356	(.290)	.293	.158	.256
	Mixture 2 ($p = .5$, $\mu_{11} = \mu_{12} = -.5$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = .5$, $\sigma_{21} = \sigma_{22} = 1.0$, $\rho_2 = .9$)						
(.000,.000)	.054	.051	.073	(.043)	.059	.050	.071
(.141,.141)	.096	.085	.140	(.089)	.099	.076	.083
(.283,.283)	.181	.172	.278	(.219)	.231	.163	.207
(.424,.424)	.345	.311	.507	(.414)	.409	.300	.392
	Mixture 3 ($p = .5$, $\mu_{11} = \mu_{12} = -1.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = -.9$, $\mu_{21} = \mu_{22} = 1.0$, $\sigma_{21} = \sigma_{22} = 1.0$, $\rho_2 = 0.0$)						
(.000,.000)	.050	.045	.089	(.062)	.086	.050	.092
(.141,.141)	.058	.068	.106	(.070)	.110	.051	.060
(.283,.283)	.086	.106	.199	(.157)	.235	.115	.135
(.424,.424)	.136	.168	.390	(.298)	.424	.191	.268
	Mixture 4 ($p = .9$, $\mu_{11} = \mu_{22} = 0.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = 0.0$, $\sigma_{21} = \sigma_{22} = 20.0$, $\rho_2 = 0.0$)						
(.000,.000)	.037	.039	.022	(.012)	.044	.035	.033
(.141,.141)	.092	.085	.050	(.024)	.102	.067	.083
(.283,.283)	.261	.232	.120	(.088)	.260	.198	.250
(.424,.424)	.500	.466	.231	(.193)	.494	.402	.469

Table 4.13--continued.

Bivariate medians	Statistics B	H	T^2	MB	B1	B2
	Mixture 5 ($p = .9$, $\mu_{11} = \mu_{12} = 0.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = 0.0$, $\sigma_{21} = 20.0$, $\sigma_{22} = 40.0$, $\rho_2 = 0.0$)					
(.000,.000)	.037	.039	.018 (.008)	.051	.035	.032
(.141,.141)	.087	.088	.040 (.021)	.097	.067	.081
(.283,.283)	.251	.230	.107 (.079)	.268	.197	.248
(.424,.424)	.502	.467	.191 (.157)	.485	.401	.468

Entries: Proportion of times each test statistic exceeded its asymptotic critical value. Values in parentheses are the rejection proportions for Hotelling's T^2 using the $F_{2,n-2}$ distribution.

Nominal significance level: .05318.

Number of samples: 1000 from each distribution.

Sample size: 25.

Distributions: Bivariate normal mixture, choosing N_1 with probability p and N_2 with probability 1-p, where N_i, $i = 1, 2$, is bivariate normal with null mean (μ_{i1}, μ_{i2}) and covariance matrix

$$\begin{pmatrix} \sigma_{i1}^2 & \rho_i\sigma_{i1}\sigma_{i2} \\ \rho_i\sigma_{i1}\sigma_{i2} & \sigma_{i2}^2 \end{pmatrix}.$$

are heavy-tailed distributions. In each case a bivariate normal random variable with independent standard normal components is contaminated with a bivariate normal variable with independent components, each having a mean equal to zero but a relatively large variance. Again the superiority of MB and the other competitors over Hotelling's T^2 is demonstrated for heavy-tailed distributions.

Lastly, we note that we attempted to approximate the null distribution of MB for small samples using the null distribution of

$$MB^* = 6n^{-1}(C_{MB}^{*2}+S_{MB}^{*2}),$$

where

$$C_{MB}^* = \frac{1}{n}\sum_{i=1}^{n} a_i \cos\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)(\text{Rank}(D_i^*)),$$

$$S_{MB}^* = \frac{1}{n}\sum_{i=1}^{n} a_i \sin\left(\frac{\pi \text{Rank}(\phi_i)}{n}\right)(\text{Rank}(D_i^*)),$$

and D_i^* is as in Chapter 3. That is, we replaced D_i by D_i^* in the definition of the test statistic (see 2.1.5 and 3.1.2). The independence of ϕ_i and D_i^* under the null hypothesis allows us to obtain the null distribution of MB^* by generating all possible assignments of ranks to $\phi_1,\ldots,\phi_n$ and $D_1^*,\ldots,D_n^*$. For larger samples, this is quite computer-intensive, so we instead randomly selected 100,000 rank assignments from all possible assignments of the ranks to $\phi_1,\ldots,\phi_n$ and $D_1^*,\ldots,D_n^*$. Monte Carlo simulations were then performed using the approximate percentiles obtained. The results

were inconclusive. In many cases these small sample percentiles appeared closer to the true percentiles than that obtained from the asymptotic distribution; however, this was not always true. In general, though, for moderately large samples, namely $n \geq 10$, the large sample percentiles appear (in Tables 4.3 - 4.13) to be reasonably close to the true percentiles of the null distribution.

In summary, of the affine-invariant statistics, Blumen's test has the greatest power for heavy-tailed distributions while MB has the greatest power for light-tailed distributions. Unlike Blumen's statistic and Hodges' statistic, MB performs very well relative to Hotelling's T^2 under bivariate normality. The only setting for which Blumen's statistic and Hodges' statistic perform better than Hotelling's T^2 is when the underlying distribution is heavy-tailed. On the other hand, MB performs better than Hotelling's T^2 for both light- and heavy-tailed distributions. Further, MB appears robust with respect to moderate departures in the assumption of elliptical symmetry. The wide range of distributions for which MB has relatively high power suggests that MB is a promising procedure for the one-sample bivariate location problem.

CHAPTER 5
A BIVARIATE TWO-SAMPLE TEST FOR LOCATION

5.1 Definition of the Test Statistic

A natural extension of the one-sample statistic, MB, introduced in Chapter 2, will be described in this section. Assume that $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$ and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independent random samples from bivariate populations with continuous distribution functions $G_1(u_1, u_2)$ and $G_2(u_1, u_2) = G_1(u_1-\theta_1, u_2-\theta_2)$, respectively. We are interested in testing

$$H_o: \underset{\sim}{\theta} = (\theta_1, \theta_2)' = \underset{\sim}{0} \quad \text{against} \quad H_a: \underset{\sim}{\theta} = (\theta_1, \theta_2)' \neq \underset{\sim}{0}. \quad (5.1.1)$$

We mentioned in the introduction that Mardia's two-sample statistic extends Blumen's work to two samples. Mardia uses the center of gravity of the standardized versions of the centered vectors from the first sample. Specifically, define

$$\underset{\sim}{\tilde{Z}} = \frac{1}{m+n}\left\{\sum_{i=1}^{m} \underset{\sim}{X}_i + \sum_{j=1}^{n} \underset{\sim}{Y}_j\right\}, \quad (5.1.2)$$

the mean vector for the combined sample. Let $\phi_{X,i}$ ($\phi_{Y,j}$) denote the angle between $\underset{\sim}{X}_i-\underset{\sim}{\tilde{Z}}$ ($\underset{\sim}{Y}_j-\underset{\sim}{\tilde{Z}}$) and the positive half of the horizontal axis, where $0 \leq \phi_{X,i}$ ($\phi_{Y,j}$) $< 2\pi$, $i = 1,\ldots,m$, $j = 1,\ldots,n$. The center of gravity (C_M, S_M) of Mardia's statistic is defined by

$$C_M = \frac{1}{m} \sum_{i=1}^{m} \cos\left(\frac{2\pi \text{Rank}(\phi_{X,i})}{N}\right)$$

and (5.1.3)

$$S_M = \frac{1}{m} \sum_{i=1}^{m} \sin\left(\frac{2\pi \text{Rank}(\phi_{X,i})}{N}\right),$$

where $N = m+n$, and $\text{Rank}(\phi_{X,i})$ denotes the rank of $\phi_{X,i}$ among $\phi_{X,1},\ldots,\phi_{X,m},\phi_{Y,1},\ldots,\phi_{Y,n}$. The sign statistic of Mardia is then

$$M = 2\frac{m}{n}(N-1)\{C_M^2+S_M^2\}. \tag{5.1.4}$$

The signed-rank statistic we propose uses the distances of the centered observations from the origin, in addition to the directions of the vectors from the origin. Define

$$\hat{\Sigma} = \frac{1}{N}\{\sum_{i=1}^{m}(\underset{\sim}{X}_i-\bar{\underset{\sim}{Z}})(\underset{\sim}{X}_i-\bar{\underset{\sim}{Z}})' + \sum_{j=1}^{n}(\underset{\sim}{Y}_j-\bar{\underset{\sim}{Z}})(\underset{\sim}{Y}_j-\bar{\underset{\sim}{Z}})'\},$$

$$D_{X,i} = (\underset{\sim}{X}_i-\bar{\underset{\sim}{Z}})'\hat{\Sigma}^{-1}(\underset{\sim}{X}_i-\bar{\underset{\sim}{Z}}),\ i = 1,\ldots,m, \tag{5.1.5}$$

$$D_{Y,j} = (\underset{\sim}{Y}_j-\bar{\underset{\sim}{Z}})'\hat{\Sigma}^{-1}(\underset{\sim}{Y}_j-\bar{\underset{\sim}{Z}}),\ j = 1,\ldots,n,$$

and let $\text{Rank}(D_{X,i})$ ($\text{Rank}(D_{Y,j})$) denote the rank of $D_{X,i}$ ($D_{Y,j}$) among $D_{X,1},\ldots,D_{X,m},D_{Y,1},\ldots,D_{Y,n}$. Let

$$C_{MB2} = \frac{1}{m}\sum_{i=1}^{m}\cos\left(\frac{2\pi \text{Rank}(\phi_{X,i})}{N}\right)\left(\frac{\text{Rank}(D_{X,i})}{N}\right)$$

$$-\frac{1}{n}\sum_{j=1}^{n}\cos\left(\frac{2\pi \text{Rank}(\phi_{Y,j})}{N}\right)\left(\frac{\text{Rank}(D_{Y,j})}{N}\right)$$

and (5.1.6)

$$S_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \sin\left(\frac{2\pi \text{Rank}(\phi_{X,i})}{N}\right)\left(\frac{\text{Rank}(D_{X,i})}{N}\right) - \frac{1}{n}\sum_{j=1}^{n} \sin\left(\frac{2\pi \text{Rank}(\phi_{Y,j})}{N}\right)\left(\frac{\text{Rank}(D_{Y,j})}{N}\right).$$

The proposed two-sample statistic is

$$MB2 = 6\left(\frac{1}{m} + \frac{1}{n}\right)^{-1}(C_{MB2}^2 + S_{MB2}^2). \tag{5.1.7}$$

Notice that, unlike the components of Mardia's statistic (in 5.1.3), C_{MB2} and S_{MB2} have each been constructed to have an asymptotic mean of zero under H_o. This centering is useful when developing asymptotic properties of MB2. Likewise, it will be convenient to examine a "centered" version of Mardia's statistic. Let

$$C_{M2} = \frac{1}{m}\sum_{i=1}^{m} \cos\left(\frac{2\pi \text{Rank}(\phi_{X,i})}{N}\right) - \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{2\pi \text{Rank}(\phi_{Y,j})}{N}\right),$$

$$S_{M2} = \frac{1}{m}\sum_{i=1}^{m} \sin\left(\frac{2\pi \text{Rank}(\phi_{X,i})}{N}\right) - \frac{1}{n}\sum_{j=1}^{n} \sin\left(\frac{2\pi \text{Rank}(\phi_{Y,j})}{N}\right),$$

and (5.1.8)

$$M2 = 2\left(\frac{1}{m} + \frac{1}{n}\right)^{-1}\{C_{M2}^2 + S_{M2}^2\}.$$

This centered modification, M2, of Mardia's two-sample statistic will be investigated and compared to MB2 in the following chapters.

5.2 Affine-Invariance of MB2

Like MB, the statistic MB2 is invariant with respect to nonsingular linear transformations of the observations. In verifying this, we use the fact that for all i and j,

$$(L\underset{\sim}{X}_i - L\bar{\underset{\sim}{Z}})'(L\hat{\Sigma}L')^{-1}(L\underset{\sim}{X}_i - L\bar{\underset{\sim}{Z}}) = (\underset{\sim}{X}_i - \bar{\underset{\sim}{Z}})'\hat{\Sigma}^{-1}(\underset{\sim}{X}_i - \bar{\underset{\sim}{Z}}) = D_{X,i}$$

and (5.2.1)

$$(L\underset{\sim}{Y}_j - L\bar{\underset{\sim}{Z}})'(L\hat{\Sigma}L')^{-1}(L\underset{\sim}{Y}_j - L\bar{\underset{\sim}{Z}}) = (\underset{\sim}{Y}_j - \bar{\underset{\sim}{Z}})'\hat{\Sigma}^{-1}(\underset{\sim}{Y}_j - \bar{\underset{\sim}{Z}}) = D_{Y,j},$$

for any 2×2 nonsingular matrix, L. That is, the standardized distances $D_{X,1},\ldots,D_{X,m}$ and $D_{Y,1},\ldots,D_{Y,n}$ are themselves invariant with respect to nonsingular linear transformations of the observations. Again, we prove the invariance of MB2 through a sequence of three lemmas.

Lemma 5.2.2. The statistic, MB2, is invariant with respect to a rotation of all of the observations.

Proof. Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$, $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$, $\phi_{X,1},\ldots,\phi_{X,m}$, $\phi_{Y,1},\ldots,\phi_{Y,n}$, and $D_{X,1},\ldots,D_{X,m}$, $D_{Y,1},\ldots,D_{Y,n}$ be as already defined. Now rotate the observations clockwise, yielding the transformed vectors, $\underset{\sim}{X}_1^* = P\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m^* = P\underset{\sim}{X}_m$, and $\underset{\sim}{Y}_1^* = P\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n^* = P\underset{\sim}{Y}_n$, where P is a 2×2 orthogonal matrix of the type discussed in the proof of Theorem 2.2.7. If we let

$$\bar{\underset{\sim}{Z}}^* = \frac{1}{N}\{\sum_{i=1}^{m}\underset{\sim}{X}_i^* + \sum_{j=1}^{n}\underset{\sim}{Y}_j^*\},$$

then $\bar{Z}^* = P\bar{Z}$. Hence if we find the angles $\phi^*_{X,1},\ldots,\phi^*_{X,m}$ and $\phi^*_{Y,1},\ldots,\phi^*_{Y,n}$ associated with $X^*_1,\ldots,X^*_m$ and $Y^*_1,\ldots,Y^*_n$ to construct MB2, then $\phi^*_{X,i}$ ($\phi^*_{Y,j}$) represents the angle between $P(X_i-\bar{Z})$ ($P(Y_j-\bar{Z})$) and the positive half of the horizontal axis. The vectors $P(X_1-\bar{Z}),\ldots,P(X_m-\bar{Z})$ and $P(Y_1-\bar{Z}),\ldots,P(Y_m-\bar{Z})$ are obtained from $X_1-\bar{Z},\ldots,X_m-\bar{Z}$ and $Y_1-\bar{Z},\ldots,Y_n-\bar{Z}$, respectively, by rotating each vector clockwise via P. Assume that only one centered vector is rotated across the horizontal axis. Without loss in generality, assume that this vector is $X_k-\bar{Z}$. Then

$$\text{Rank}(\phi^*_{X,i}) = \begin{cases} \text{Rank}(\phi_{X,i})-1, & \text{if } i \neq k, \\ N, & \text{if } i = k, \end{cases}$$

and

$$\text{Rank}(\phi^*_{Y,j}) = \text{Rank}(\phi_{Y,j}) - 1, \quad \text{for all } j.$$

We now construct MB2 from the rotated observations, yielding

$$C^*_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \cos\left(\frac{2\pi\,\text{Rank}(\phi^*_{X,i})}{N}\right)\left(\frac{\text{Rank}(D_{X,i})}{N}\right)$$

$$- \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{2\pi\,\text{Rank}(\phi^*_{Y,j})}{N}\right)\left(\frac{\text{Rank}(D_{Y,j})}{N}\right),$$

$$S^*_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \sin\left(\frac{2\pi \mathrm{Rank}(\phi^*_{X,i})}{N}\right)\left(\frac{\mathrm{Rank}(D_{X,i})}{N}\right)$$

$$- \frac{1}{n}\sum_{j=1}^{n} \sin\left(\frac{2\pi \mathrm{Rank}(\phi^*_{Y,j})}{N}\right)\left(\frac{\mathrm{Rank}(D_{Y,j})}{N}\right),$$

and

$$MB2 = 6\left(\frac{1}{m} + \frac{1}{n}\right)^{-1}\{C^*_{MB2} + S^*_{MB2}\},$$

using 5.1.6, 5.1.7, and 5.2.1. Noting that

$$\cos\left(\frac{2\pi \mathrm{Rank}(\phi^*_{X,k})}{N}\right) = \cos\left(\frac{2\pi N}{N}\right) = \cos(0) = \cos\left(\frac{2\pi(\mathrm{Rank}(\phi_{X,k})-1)}{N}\right),$$

and

$$\sin\left(\frac{2\pi \mathrm{Rank}(\phi^*_{X,k})}{N}\right) = \sin\left(\frac{2\pi(\mathrm{Rank}(\phi_{X,k})-1)}{N}\right),$$

We can replace $\mathrm{Rank}(\phi^*_{X,i})$ and $\mathrm{Rank}(\phi^*_{Y,j})$ by $\mathrm{Rank}(\phi_{X,i})-1$ and $\mathrm{Rank}(\phi_{Y,j})-1$, respectively, for all i and j in the expressions for C^*_{MB2} and S^*_{MB2}, yielding

$$C^*_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \cos\left(\frac{2\pi(\mathrm{Rank}(\phi_{X,i})-1)}{N}\right)\left(\frac{\mathrm{Rank}(D_{X,i})}{N}\right)$$

$$- \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{2\pi(\mathrm{Rank}(\phi_{Y,j})-1)}{N}\right)\left(\frac{\mathrm{Rank}(D_{Y,j})}{N}\right)$$

and (5.2.3)

$$S^{*}_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \sin\left(\frac{2\pi(\text{Rank}(\phi_{X,i})-1)}{N}\right)\left(\frac{\text{Rank}(D_{X,i})}{N}\right)$$

$$- \frac{1}{n}\sum_{j=1}^{n} \sin\left(\frac{2\pi(\text{Rank}(\phi_{Y,j})-1)}{N}\right)\left(\frac{\text{Rank}(D_{Y,j})}{N}\right).$$

Using the trigonometric identities given in 2.2.4, we can write

$$C^{*}_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \left\{\cos\left(\frac{2\pi\text{Rank}(\phi_{X,i})}{N}\right)\cos\left(\frac{2\pi}{N}\right)+\sin\left(\frac{2\pi\text{Rank}(\phi_{X,i})}{N}\right)\sin\left(\frac{2\pi}{N}\right)\right\}\frac{\text{Rank}(D_{X,i})}{N}$$

$$- \frac{1}{n}\sum_{j=1}^{n} \left\{\cos\left(\frac{2\pi\text{Rank}(\phi_{Y,j})}{N}\right)\cos\left(\frac{2\pi}{N}\right)+\sin\left(\frac{2\pi\text{Rank}(\phi_{Y,j})}{N}\right)\sin\left(\frac{2\pi}{N}\right)\right\}\frac{\text{Rank}(D_{Y,j})}{N}$$

and

$$S^{*}_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \left\{\sin\left(\frac{2\pi\text{Rank}(\phi_{X,i})}{N}\right)\cos\left(\frac{2\pi}{N}\right)-\cos\left(\frac{2\pi\text{Rank}(\phi_{X,i})}{N}\right)\sin\left(\frac{2\pi}{N}\right)\right\}\frac{\text{Rank}(D_{X,i})}{N}$$

$$- \frac{1}{n}\sum_{j=1}^{n} \left\{\sin\left(\frac{2\pi\text{Rank}(\phi_{Y,j})}{N}\right)\cos\left(\frac{2\pi}{N}\right)-\cos\left(\frac{2\pi\text{Rank}(\phi_{Y,j})}{N}\right)\sin\left(\frac{2\pi}{N}\right)\right\}\frac{\text{Rank}(D_{Y,j})}{N}.$$

That is,

$$C^{*}_{MB2} = \cos\left(\frac{2\pi}{N}\right)C_{MB2} + \sin\left(\frac{2\pi}{N}\right)S_{MB2}$$

and

$$S^{*}_{MB2} = \cos\left(\frac{2\pi}{N}\right)S_{MB2} - \sin\left(\frac{2\pi}{N}\right)C_{MB2}.$$

Hence,

$$C^{*2}_{MB2}+S^{*2}_{MB2} = (\cos^2(\tfrac{2\pi}{N})+\sin^2(\tfrac{2\pi}{N}))\{C^2_{MB2}+S^2_{MB2}\}$$

$$= C^2_{MB2}+S^2_{MB2}.$$

As in the one-sample setting, the invariance with respect to counter-clockwise rotations and further clockwise rotations follows. □

Lemma 5.2.4. The statistic, MB2, is invariant with respect to reflections of the observations about the horizontal axis.

Proof. Let $\underset{\sim}{X}^*_1,\ldots,\underset{\sim}{X}^*_m$ and $\underset{\sim}{Y}^*_1,\ldots,\underset{\sim}{Y}^*_n$ be the reflected vectors. That is,

$$\underset{\sim}{X}^*_i = R\underset{\sim}{X}_i,\ i = 1,\ldots,m,$$

and

$$\underset{\sim}{Y}^*_j = R\underset{\sim}{Y}_j,\ j = 1,\ldots,n,$$

where

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\underset{\sim}{\bar{Z}}^* = \frac{1}{N}\{\sum_{i=1}^{m}\underset{\sim}{X}^*_i + \sum_{j=1}^{n}\underset{\sim}{Y}^*_j\} = R\underset{\sim}{\bar{Z}},$$

$$\underset{\sim}{X}^*_i-\underset{\sim}{\bar{Z}}^* = R(\underset{\sim}{X}_i-\underset{\sim}{\bar{Z}}),$$

and

$$\underset{\sim}{Y}^*_j-\underset{\sim}{\bar{Z}}^* = R(\underset{\sim}{Y}_j-\underset{\sim}{\bar{Z}}).$$

So, the centered vectors are also reflected about the horizontal axis. Letting $\phi^*_{X,i}$ ($\phi^*_{Y,j}$) denote the angle between $\underset{\sim}{X}^*_i - \bar{\underset{\sim}{Z}}^*$ ($\underset{\sim}{Y}^*_j - \bar{\underset{\sim}{Z}}^*$) and the positive half of the horizontal axis, it follows that the ordering of the angles $\phi^*_{X,1},\ldots,\phi^*_{X,m}$, $\phi^*_{Y,1},\ldots,\phi^*_{Y,n}$ is the reverse of the ordering of the angles $\phi_{X,1},\ldots,\phi_{X,m}$, $\phi_{Y,1},\ldots,\phi_{Y,n}$. That is,

$$\text{Rank}(\phi^*_{X,i}) = N+1-\text{Rank}(\phi_{X,i}),$$

and

$$\text{Rank}(\phi^*_{Y,j}) = N+1-\text{Rank}(\phi_{Y,j}),$$

for all i and j. As in 5.2.3, we compute C^*_{MB2} and S^*_{MB2} from the reflected observations and find that

$$C^*_{MB2}$$

$$= \frac{1}{m}\sum_{i=1}^{m} \cos\left(\frac{2\pi\text{Rank}(\phi^*_{X,i})}{N}\right)\left(\frac{\text{Rank}(D_{X,i})}{N}\right) - \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{2\pi\text{Rank}(\phi^*_{Y,j})}{N}\right)\left(\frac{\text{Rank}(D_{Y,j})}{N}\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m} \cos\left(\frac{2\pi(N+1)}{N} - \frac{2\pi\text{Rank}(\phi_{X,i})}{N}\right)\left(\frac{\text{Rank}(D_{X,i})}{N}\right)$$

$$- \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{2\pi(N+1)}{N} - \frac{2\pi\text{Rank}(\phi_{Y,j})}{N}\right)\left(\frac{\text{Rank}(D_{Y,j})}{N}\right)$$

$$= \cos\left(\frac{2\pi(N+1)}{N}\right)C_{MB2} + \sin\left(\frac{2\pi(N+1)}{N}\right)S_{MB2},$$

using the identities in 2.2.4. Also

$$S^*_{MB2} = \sin(\frac{2\pi(N+1)}{N})C_{MB2} - \cos(\frac{2\pi(N+1)}{N})S_{MB2}.$$

So,

$$C^{*2}_{MB2} + S^{*2}_{MB2} = C^2_{MB2} + S^2_{MB2}. \square$$

Lemma 5.2.5. The statistic, MB2, is invariant with respect to changes in scale.

Proof. Apply the proof of Lemma 2.2.6 to the vectors $\underset{\sim}{X}_1 - \bar{\underset{\sim}{Z}}, \ldots, \underset{\sim}{X}_m - \bar{\underset{\sim}{Z}}, \underset{\sim}{Y}_1 - \bar{\underset{\sim}{Z}}, \ldots, \underset{\sim}{Y}_n - \bar{\underset{\sim}{Z}}$. $\square$

Theorem 5.2.6. The statistic, MB2, is invariant with respect to nonsingular linear transformations of the observations.

Proof. Using Lemmas 5.2.2, 5.2.4, and 5.2.5, the proof follows the pattern of Theorem 2.2.7. $\square$

It is also important for the two-sample problem to show that MB2 is invariant with respect to translations of the observations. That is,

$$MB2(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m, \underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n) = MB2(\underset{\sim}{X}^*_1, \ldots, \underset{\sim}{X}^*_m, \underset{\sim}{Y}^*_1, \ldots, \underset{\sim}{Y}^*_n),$$

where $\underset{\sim}{X}^*_i = \underset{\sim}{X}_i + \underset{\sim}{d}$, $\underset{\sim}{Y}^*_j = \underset{\sim}{Y}_j + \underset{\sim}{d}$ for all i and j, and $\underset{\sim}{d} = (d_1, d_2)'$ is a constant vector. Note that MB2 is a function only of $\underset{\sim}{X}_1 - \bar{\underset{\sim}{Z}}, \ldots, \underset{\sim}{X}_m - \bar{\underset{\sim}{Z}}$, $\underset{\sim}{Y}_1 - \bar{\underset{\sim}{Z}}, \ldots, \underset{\sim}{Y}_n - \bar{\underset{\sim}{Z}}$,

$$\bar{\underset{\sim}{Z}}^* = \frac{1}{N}\{\sum_{i=1}^{m} \underset{\sim}{X}^*_i + \sum_{j=1}^{n} \underset{\sim}{Y}^*_j\} = \bar{\underset{\sim}{Z}} + \underset{\sim}{d},$$

$$\underset{\sim}{X}_i^* - \bar{\underset{\sim}{Z}}^* = (\underset{\sim}{X}_i - \underset{\sim}{d}) - (\bar{\underset{\sim}{Z}} + \underset{\sim}{d}) = \underset{\sim}{X}_i - \bar{\underset{\sim}{Z}},$$

and

$$\underset{\sim}{Y}_j^* - \bar{\underset{\sim}{Z}}^* = \underset{\sim}{Y}_j - \bar{\underset{\sim}{Z}},$$

for all i and j. Hence the translation invariance follows.

Lemma 5.2.7. The statistic, MB2, is invariant with respect to translations of the observations.

Slight modifications in the proofs of this section will demonstrate that the two versions of Mardia's statistic (given by 5.1.4 and 5.1.8) are also affine-invariant.

CHAPTER 6
ASYMPTOTICS FOR THE TWO-SAMPLE PROBLEM

6.1 Asymptotic Null Distribution of MB2

In this section, we find the asymptotic null distribution of MB2 under the assumption that the underlying populations are elliptically symmetric (see 3.1.14). As in the one-sample case, we introduce a statistic which has the same asymptotic null distribution as MB2. Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$ and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be independent random samples from continuous elliptically symmetric bivariate distribution functions, $G_1(u_1, u_2)$ and $G_2(u_1, u_2) = G_1(u_1-\theta_1, u_2-\theta_2)$, respectively. Note, under H_o: $\underset{\sim}{\theta} = (\theta_1, \theta_2)' = \underset{\sim}{0}$, $\underset{\sim}{X}_i = \underset{\sim}{Y}_j$, for all i and j. Let

$$D^*_{X,i} = (\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(\underset{\sim}{X}_i-\underset{\sim}{\mu}),\ i = 1,\ldots,m,$$

and (6.1.1)

$$D^*_{Y,j} = (\underset{\sim}{Y}_j-\underset{\sim}{\mu})'\Sigma_2^{-1}(\underset{\sim}{Y}_j-\underset{\sim}{\mu}),\ j = 1,\ldots,n,$$

where

$$\underset{\sim}{\mu} = E_{H_o}[\underset{\sim}{X}_i] = E_{H_o}[\underset{\sim}{Y}_j],$$

and (6.1.2)

$$\Sigma_2 = E_{H_o}[(\underset{\sim}{X}_i-\underset{\sim}{\mu})(\underset{\sim}{X}_i-\underset{\sim}{\mu})'] = E_{H_o}[(\underset{\sim}{Y}_j-\underset{\sim}{\mu})(\underset{\sim}{Y}_j-\underset{\sim}{\mu})'].$$

Let

$$F_2(t) = P_{H_o}(D^*_{X,i} \le t) = P_{H_o}(D^*_{Y,j} \le t), \tag{6.1.3}$$

and for $i = 1,\ldots,m$ and $j = 1,\ldots,n$ define $\phi^*_{X,i}$ ($\phi^*_{Y,j}$) to be the angle between $\underset{\sim}{X}_i - \underset{\sim}{\mu}$ ($\underset{\sim}{Y}_j - \underset{\sim}{\mu}$) and the positive half of the horizontal axis, $0 \le \phi^*_{X,i}$ ($\phi^*_{Y,j}$) $< 2\pi$. Let

$$H_2(t) = P_{H_o}(\phi^*_{X,i} \le t) = P_{H_o}(\phi^*_{Y,j} \le t). \tag{6.1.4}$$

Finally, define two empirical distribution functions

$$F_N(t) = \frac{1}{N}\{\sum_{i=1}^{m} I[D_{X,i} \le t] + \sum_{j=1}^{n} I[D_{Y,j} \le t]\},$$

and (6.1.5)

$$H_N(t) = \frac{1}{N}\{\sum_{i=1}^{m} I[\phi_{X,i} \le t] + \sum_{j=1}^{n} I[\phi_{Y,j} \le t]\},$$

where $D_{X,i}$, $D_{Y,j}$, $\phi_{X,i}$ and $\phi_{Y,j}$ are as given in Chapter 5.

Using 5.1.6, we can write

$$C_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \cos(2\pi H_N(\phi_{X,i}))F_N(D_{X,i}) - \frac{1}{n}\sum_{j=1}^{n} \cos(2\pi H_N(\phi_{Y,j})F_N(D_{Y,j})$$

and (6.1.6)

$$S_{MB2} = \frac{1}{m}\sum_{i=1}^{m} \sin(2\pi H_N(\phi_{X,i}))F_N(D_{X,i}) - \frac{1}{n}\sum_{j=1}^{n} \sin(2\pi H_N(\phi_{Y,j}))F_N(D_{Y,j}).$$

Define

$$\bar{C}_{MB2} = \frac{1}{m}\sum_{i=1}^{m}\cos(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i}) - \frac{1}{n}\sum_{j=1}^{n}\cos(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j})$$

and (6.1.7)

$$\bar{S}_{MB2} = \frac{1}{m}\sum_{i=1}^{m}\sin(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i}) - \frac{1}{n}\sum_{j=1}^{n}\sin(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j}).$$

We will show that $(\frac{1}{m}+\frac{1}{n})^{-1/2}(C_{MB2}, S_{MB2})$ and $(\frac{1}{m}+\frac{1}{n})^{-1/2}(\bar{C}_{MB2}, \bar{S}_{MB2})$ have the same asymptotic null distribution. To do this we will need to show that

$$|H_N(\phi_{X,i}) - H_2(\phi^*_{X,i})| \overset{H_o}{=} o_p(1), \tag{6.1.8}$$

$$|H_N(\phi_{Y,j}) - H_2(\phi^*_{Y,j})| \overset{H_o}{=} o_p(1), \tag{6.1.9}$$

$$|F_N(D_{X,i}) - F_2(D^*_{X,i})| \overset{H_o}{=} o_p(1), \tag{6.1.10}$$

and

$$|F_N(D_{Y,j}) - F_2(D^*_{Y,j})| \overset{H_o}{=} o_p(1). \tag{6.1.11}$$

Since under H_o, $\underset{\sim}{X}_i \overset{d}{=} \underset{\sim}{Y}_j$ for all i and j, 6.1.8 is equivalent to 6.1.9 and 6.1.10 is equivalent to 6.1.11. Hence we will verify explicitly 6.1.8 and 6.1.10. Using Lemma 3.1.4, it will suffice to establish

$$N^{-1/2}(\bar{Z}-\underset{\sim}{\mu}) \overset{H_o}{=} o_p(1), \tag{6.1.12}$$

$$N^{-1/2}(\hat{\Sigma}^{-1}-\Sigma_2^{-1}) \overset{H_o}{=} o_p(1), \tag{6.1.13}$$

$$\sup_{\substack{r\in B_1 \\ w\in B_2}} \left|(X_i-(\underset{\sim}{\mu}+N^{-\frac{1}{2}}\underset{\sim}{r}))'(\Sigma_2^{-1}+N^{-\frac{1}{2}}w)(X_i-(\underset{\sim}{\mu}+N^{-\frac{1}{2}}\underset{\sim}{r}))-(X_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(X_i-\underset{\sim}{\mu})\right| \overset{H_o}{=} o_p(1), \tag{6.1.14}$$

and

$$\sup_{r\in B_1} \left|\phi^*_{X,i}(\underset{\sim}{\mu}+N^{-\frac{1}{2}}\underset{\sim}{r})-\phi^*_{X,i}(\underset{\sim}{\mu})\right| \overset{H_o}{=} o_p(1), \tag{6.1.15}$$

where $\phi^*_{X,i}(\underset{\sim}{A})$ is the angle between $X_i-\underset{\sim}{A}$ and the positive half of the horizontal axis, B_1 is a bounded sphere in R^2, and B_2 is a class of 2×2 symmetric matrices with uniformly bounded elements. (Conditions 6.1.12 and 6.1.15 establish conditions a and b, respectively, of Lemma 3.1.4, showing that 6.1.8 holds. Similarly conditions 6.1.12 and 6.1.13 jointly verify condition (a) of Lemma 3.1.4 and 6.1.14 verifies condition (b) of the lemma, establishing 6.1.10.) Conditions 6.1.12-6.1.15 are established in the following lemma.

Lemma 6.1.16. Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$, $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be i.i.d. as $\underset{\sim}{Z}$, where $\underset{\sim}{Z} = (Z_1, Z_2)'$ is from a bivariate continuous distribution function with finite mean vector $\underset{\sim}{\mu}$ and positive definite covariance matrix Σ_2. Let $\bar{\underset{\sim}{Z}}$, $\hat{\Sigma}$, and $\phi^*_{X,i}(A)$ be as defined previously. Assuming the covariance matrix of $(Z_1, Z_2, Z_1^2, Z_2^2, Z_1Z_2)$ is positive definite, conditions 6.1.12 through 6.1.15 hold.

Proof. Condition 6.1.12 is immediate from the central limit theorem. We use Lemma 3.1.10 to show that 6.1.13 holds. Let $N = m+n$,

$$\hat{\Sigma} = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12} & \Delta_{22} \end{pmatrix}, \tag{6.1.17}$$

$\underset{\sim}{S} = (s_{11}, s_{22}, s_{12})'$, and $\underset{\sim}{\Delta} = (\Delta_{11}, \Delta_{22}, \Delta_{12})'$.

We first verify that $N^{1/2}(\underset{\sim}{S}-\underset{\sim}{\Delta})$ is asymptotically trivariate normal. Let

$$\underset{\sim}{V} = (V_1, V_2, V_3, V_4, V_5)'$$

$$= N^{-1}\Big(\sum_{i=1}^{m} X_{1i} + \sum_{j=1}^{n} Y_{1j}, \ \sum_{i=1}^{m} X_{2i} + \sum_{j=1}^{n} Y_{2j}, \ \sum_{i=1}^{m} X_{1i}X_{2i} + \sum_{j=1}^{n} Y_{1j}Y_{2j},$$

$$\sum_{i=1}^{m} X_{1i}^2 + \sum_{j=1}^{n} Y_{1j}^2, \ \sum_{i=1}^{m} X_{2i}^2 + \sum_{j=1}^{n} Y_{2j}^2\Big)',$$

where

$$\underset{\sim}{X}_i = (X_{1i}, X_{2i})' \qquad i = 1,\ldots,m$$

and

$$\underset{\sim}{Y}_j = (Y_{1j}, Y_{2j})' \qquad j = 1,\ldots,n.$$

By the Multivariate Central Limit Theorem (recalling $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$, $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are i.i.d., here) $N^{1/2}(\underset{\sim}{V}-E[\underset{\sim}{V}])$ is asymptotically multivariate normal with mean $\underset{\sim}{0}$ and covariance matrix $\Sigma_V = \mathrm{Var}[\underset{\sim}{V}]$. Define

$$\begin{aligned}\underset{\sim}{g}(\underset{\sim}{V}) &= (g_1(\underset{\sim}{V}),\ g_2(\underset{\sim}{V}),\ g_3(\underset{\sim}{V}))' \\ &= (V_4-V_1^2,\ V_5-V_2^2,\ V_3-V_1V_2)' \\ &= (s_{11},\ s_{22},\ s_{12})',\end{aligned}$$

and let

$$\underset{\sim}{\mu}_V = (\mu_{1V},\ldots,\mu_{5V})' = E[\underset{\sim}{V}],$$

then the partial derivatives of g_1 evaluated at $\underset{\sim}{\mu}_V$ are

$$\frac{\partial g_1}{\partial V_1}\Big|_{\underset{\sim}{V}=\underset{\sim}{\mu}_V} = -2V_1\Big|_{\underset{\sim}{V}=\underset{\sim}{\mu}_V} = -2\mu_{1V},$$

$$\frac{\partial g_1}{\partial V_2}\Big|_{\underset{\sim}{V}=\underset{\sim}{\mu}_V} = \frac{\partial g_1}{\partial V_3}\Big|_{\underset{\sim}{V}=\underset{\sim}{\mu}_V} = \frac{\partial g_1}{\partial V_5}\Big|_{\underset{\sim}{V}=\underset{\sim}{\mu}_V} = 0,$$

and

$$\frac{\partial g_1}{\partial V_4}\Big|_{\underset{\sim}{V}=\underset{\sim}{\mu}_V} = 1.$$

Since these partials are continuous and not all zero at $\underset{\sim}{\mu}_V$, g_1 has a nonzero differential at $\underset{\sim}{\mu}_V$. (See remark 3.1.11.) Similarly

$$\frac{\partial g_2}{\partial v_1}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = \frac{\partial g_2}{\partial v_3}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = \frac{\partial g_2}{\partial v_4}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = 0,$$

$$\frac{\partial g_2}{\partial v_2}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = -2\mu_{2V}, \quad \frac{\partial g_2}{\partial v_5}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = 1;$$

and

$$\frac{\partial g_3}{\partial v_4}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = \frac{\partial g_3}{\partial v_5}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = 0,$$

$$\frac{\partial g_3}{\partial v_1}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = -\mu_{2V}, \quad \frac{\partial g_3}{\partial v_2}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = -\mu_{1V},$$

$$\frac{\partial g_3}{\partial v_3}\Big|_{\underset{\sim}{v}=\underset{\sim}{\mu}_V} = 1.$$

So, g_2 and g_3 each have nonzero differentials at $\underset{\sim}{\mu}_V$. Hence, by Lemma 3.1.10, $N^{1/2}(\underset{\sim}{S}-\underset{\sim}{\Delta})$ is asymptotically trivariate normal. The proof is now easily completed by referring to the proof of Lemma 3.1.12, in which we showed that $N^{1/2}(\tilde{\Sigma}^{-1}-\Sigma_o^{-1})$ is $o_p(1)$ if the elements of $N^{1/2}(\tilde{\Sigma}-\Sigma_o)$ are jointly asymptotically multivariate normal. (Given the asymptotic multivariate normality of the elements of $N^{1/2}(\tilde{\Sigma}-\Sigma_o)$, the exact structures of $\tilde{\Sigma}$ and Σ_o are unimportant, provided Σ_o is positive definite.)

In verifying condition 6.1.14, let

$$\underset{\sim}{\mu} = (\mu_1, \mu_2)', \quad \Sigma_2^{-1} = \begin{pmatrix} \Delta_{11}^* & \Delta_{12}^* \\ \Delta_{12}^* & \Delta_{22}^* \end{pmatrix},$$

$\underset{\sim}{r} = (r_1, r_2)'$, and $w = \begin{pmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{pmatrix}$.

Choose M such that $|r_i| \le M$, $i = 1,2$ and $|w_{ij}| \le M$, $1 \le i \le j \le 2$, for $\underset{\sim}{r} \in \beta_1$ and $w \in \beta_2$. Then

$$\sup_{\substack{\underset{\sim}{r}\in\beta_1 \\ w\in\beta_2}} \left|\{(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))'(\Sigma_2^{-1}+N^{-1/2}w)(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r})) - (\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(\underset{\sim}{X}_i-\underset{\sim}{\mu})\}\right|$$

$$= \sup_{\substack{\underset{\sim}{r}\in\beta_1 \\ w\in\beta_2}} \left|\{(\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(\underset{\sim}{X}_i-\underset{\sim}{\mu}) - (\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(N^{-1/2}\underset{\sim}{r}) - (N^{-1/2}\underset{\sim}{r}')\Sigma_2^{-1}(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))\right.$$

$$\left. + (\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))'N^{-1/2}w(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r})) - (\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(\underset{\sim}{X}_i-\underset{\sim}{\mu})\}\right|$$

$$= \sup_{\substack{\underset{\sim}{r}\in\beta_1 \\ w\in\beta_2}} \left|\{-(\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(N^{-1/2}\underset{\sim}{r}) - (N^{-1/2}\underset{\sim}{r}')\Sigma_2^{-1}(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))\right.$$

$$\left. + (\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))'N^{-1/2}w(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))\}\right|$$

$$\le \sup_{\substack{\underset{\sim}{r}\in\beta_1 \\ w\in\beta_2}} \left|(\underset{\sim}{X}_i-\underset{\sim}{\mu})'\Sigma_2^{-1}(N^{-1/2}\underset{\sim}{r})\right| + \sup_{\substack{\underset{\sim}{r}\in\beta_1 \\ w\in\beta_2}} \left|(N^{-1/2}\underset{\sim}{r}')\Sigma_2^{-1}(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))\right|$$

$$+ \sup_{\substack{\underset{\sim}{r}\in\beta_1 \\ w\in\beta_2}} \left|(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))'N^{-1/2}w(\underset{\sim}{X}_i-(\underset{\sim}{\mu}+N^{-1/2}\underset{\sim}{r}))\right|.$$

Considering the first term, write

$$\sup_{r\in\beta_1} |(X_i-\mu)'\Sigma_2^{-1}(N^{-1/2}r)|$$
$$w\in\beta_2$$
$$= \sup_{r\in\beta_1} N^{-1/2}|\{(X_1-\mu_1)\Delta_{11}^* r_1+(X_2-\mu_2)\Delta_{12}^* r_1+(X_1-\mu_1)\Delta_{12}^* r_2+(X_2-\mu_2)\Delta_{22}^* r_2\}|$$
$$w\in\beta_2$$
$$\leq N^{-1/2}M\{|(X_1-\mu_1)\Delta_{11}^*|+|(X_2-\mu_2)\Delta_{12}^*|+|(X_1-\mu_1)\Delta_{12}^*|+|(X_2-\mu_2)\Delta_{22}^*|\}$$
$$\xrightarrow{P} 0, \text{ as } N\to\infty.$$

Similar arguments show that the other terms also converge in probability to zero, as $N\to\infty$, establishing 6.1.14.

As for condition 6.1.15, define

$$g(h(r)) = \phi_{X,i}^*(\mu+N^{-1/2}r),$$

taking

$$h(r) = (X_{1i}-\mu_1-N^{-1/2}r_1,\ X_{2i}-\mu_2-N^{-1/2}r_2)$$

and $g(h(r))$ equal to the angle between $h(r)$ and the positive half of the horizontal axis. Now

$$\sup_{r\in\beta_1} |h(r)-h(0)|$$
$$= \sup_{r\in\beta_1} |(-N^{-1/2}r_1,\ -N^{-1/2}r_2)|$$
$$= \sup_{r\in\beta_1} N^{-1/2}(r_1^2+r_2^2)^{1/2}$$
$$\leq N^{-1/2}(2M^2)^{1/2}$$
$$\to 0, \text{ as } N\to\infty.$$

That is,

$$\sup_{r\epsilon\beta_1} |h(r)-h(0)| \xrightarrow{a.c.} 0.$$

Since g is a continuous function of h a.c., g o h is uniformly continuous a.c. for $r\epsilon\beta_1$. We have, then, that

$$\sup_{r\epsilon\beta_1} |g(h(r))-g(h(0))| \xrightarrow{a.c.} 0,$$

as $N \to \infty$. This verifies 6.1.15 and Lemma 6.1.16 is established. □

We now have the preliminary results needed to establish the asymptotic approximation.

Lemma 6.1.18. Let $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$ be independent random samples from continuous distribution functions $G_1(u_1, u_2)$ and $G_2(u_1, u_2) = G_1(u_1-\theta_1, u_2-\theta_2)$, respectively. If, under H_o: $\theta = 0$ the conditions of Lemma 6.1.16 hold and $\lim_{N\to\infty} \frac{m}{N} = \lambda$, $0 < \lambda < 1$, where $N = m+n$, then

$$(\tfrac{1}{m} + \tfrac{1}{n})^{-1/2}(C_{MB2}-\bar{C}_{MB2},\ S_{MB2}-\bar{S}_{MB2}) \xrightarrow[H_o]{P} (0,0),$$

as $N \to \infty$.

Proof. We first consider

$$E_{H_o}[\{(\frac{1}{m}+\frac{1}{n})^{-1/2}(C_{MB2}-\bar{C}_{MB2})\}^2]$$

$$= (\frac{1}{m}+\frac{1}{n})^{-1}E_{H_o}[\{[\frac{1}{m}\sum_{i=1}^{m}\cos(2\pi H_N(\phi_{X,i}))F_N(D_{X,i})-\frac{1}{n}\sum_{j=1}^{n}\cos(2\pi H_N(\phi_{Y,j}))F_N(D_{Y,j})]$$

$$-[\frac{1}{m}\sum_{i=1}^{m}\cos(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i})-\frac{1}{n}\sum_{j=1}^{n}\cos(2\pi H_2(\phi^*_{Y,j})F_2(D^*_{Y,j}))]\}^2]$$

$$= (\frac{1}{m}+\frac{1}{n})^{-1}E_{H_o}[\{\frac{1}{m}\sum_{i=1}^{m}[\cos(2\pi H_N(\phi_{X,i}))F_N(D_{X,i})-\cos(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i})]$$

$$-\frac{1}{n}\sum_{j=1}^{n}[\cos(2\pi H_N(\phi_{Y,j})F_N(D_{Y,j})-\cos(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j})]\}^2]$$

$$= (\frac{1}{m}+\frac{1}{n})^{-1}E_{H_o}[\{\frac{1}{m}\sum_{i=1}^{m}U_i-\frac{1}{n}\sum_{j=1}^{n}V_j\}^2], \qquad (6.1.19)$$

where

$$U_i = \cos(2\pi H_N(\phi_{X,i}))F_N(D_{X,i})-\cos(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i}),$$

and

$$V_j = \cos(2\pi H_N(\phi_{Y,j}))F_N(D_{Y,j})-\cos(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j}),$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Expanding the expression in 6.1.19, we can write

$$E_{H_o}[\{(\frac{1}{m}+\frac{1}{n})^{-1/2}(C_{MB2}-\bar{C}_{MB2})\}^2]$$

$$= (\frac{1}{m}+\frac{1}{n})^{-1}E_{H_o}[\{m^{-2}(\sum_{i=1}^{m}U_i)^2+n^{-2}(\sum_{j=1}^{n}V_j)^2-\frac{2}{mn}(\sum_{i=1}^{m}U_i)(\sum_{j=1}^{n}V_j)\}]$$

$$= (\frac{1}{m}+\frac{1}{n})^{-1}E_{H_o}[\{m^{-2}(\sum_{i=1}^{m}U_i^2+\sum_{i\neq k}U_iU_k)+n^{-2}(\sum_{j=1}^{n}V_j^2+\sum_{j\neq k}V_jV_k)-\frac{2}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}U_iV_j\}]$$

$$= (\frac{1}{m}+\frac{1}{n})^{-1}\{E_{H_o}[\frac{1}{m}U_i^2+\frac{m-1}{m}U_iU_k]+E_{H_o}[\frac{1}{n}V_j^2+\frac{n-1}{n}V_iV_k]-E_{H_o}[2U_iV_j]\}, \; i \neq k.$$

Since under H_o $U_1,\ldots,U_m$, $V_1,\ldots,V_n$ are exchangeable, we can simplify the above expression to find that

$$E_{H_o}[\{(\tfrac{1}{m}+\tfrac{1}{n})^{-1/2}(C_{MB2}-\bar{C}_{MB2})\}^2]$$

$$= (\tfrac{1}{m}+\tfrac{1}{n})^{-1}E_{H_o}[(\tfrac{1}{m}+\tfrac{1}{n})U_i^2-(\tfrac{1}{m}+\tfrac{1}{n})U_iU_k], \quad i \neq k$$

$$= E_{H_o}[U_i^2]-E_{H_o}[U_iU_k], \quad i \neq k.$$

Since $|U_i| \leq 2$ a.c., for all i, we can apply the Lebesgue Dominated Convergence Theorem to show that $E_{H_o}[\{(\frac{1}{m}+\frac{1}{n})^{-1/2}(C_{MB2}-\bar{C}_{MB2})\}^2]$ converges to zero as $N \to \infty$. To do so, we need to show that $U_i \xrightarrow[H_o]{p} 0$. Referring to 6.1.19, it is clear that $U_i \xrightarrow[H_o]{p} 0$ if

$$|H_N(\phi_{X,i})-H_2(\phi^*_{X,i})| \overset{H_o}{=} o_p(1)$$

and

$$|F_N(D_{X,i})-F_2(D^*_{X,i})| \overset{H_o}{=} o_p(1).$$

As we mentioned these conditions (see 6.1.8 and 6.1.10) are established via 6.1.12-6.1.15 which we verified in Lemma 6.1.16. By the Markov inequality, it follows that

$$(\tfrac{1}{m}+\tfrac{1}{n})^{-1/2}(C_{MB2}-\bar{C}_{MB2}) \overset{H_o}{=} o_p(1).$$

Similarly,

$$(\tfrac{1}{m}+\tfrac{1}{n})^{-1/2}(S_{MB2}-\bar{S}_{MB2}) \overset{H_o}{=} o_p(1). \quad \square$$

Lemma 6.1.18 readily yields the asymptotic null distribution of MB2 as given in the following theorem.

Theorem 6.1.20. If the conditions of Lemma 6.1.18 hold and the distribution functions G_1 and G_2 are elliptically symmetric (see 3.1.14), then MB2 is asymptotically Chi-square with two degrees of freedom under H_o.

Proof. Using Lemma 6.1.18, we need only find the asymptotic null distribution of the statistic $6(\frac{1}{m} + \frac{1}{n})^{-1}(\bar{C}^2_{MB2} + \bar{S}^2_{MB2})$. Let

$$T_N = \ell_1(6(\tfrac{1}{m} + \tfrac{1}{n})^{-1})^{1/2}\bar{C}_{MB2} + \ell_2(6(\tfrac{1}{m} + \tfrac{1}{n})^{-1})^{1/2}\bar{S}_{MB2},$$

be an arbitrary linear combination of $(6(\frac{1}{m} + \frac{1}{n})^{-1})^{1/2}\bar{C}_{MB2}$ and $(6(\frac{1}{m} + \frac{1}{n})^{-1})^{1/2}\bar{S}_{MB2}$. We will show that T_N is asymptotically normal under H_o. Write

$$\bar{C}_{MB2} = \frac{1}{m}\sum_{i=1}^{m}\bar{C}_{X,i} - \frac{1}{n}\sum_{j=1}^{n}\bar{C}_{Y,j}$$

and

$$\bar{S}_{MB2} = \frac{1}{m}\sum_{i=1}^{m}\bar{S}_{X,i} - \frac{1}{n}\sum_{j=1}^{n}\bar{S}_{Y,j},$$

where, for all i and j, (6.1.21)

$$\bar{C}_{X,i} = \cos(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i}),$$

$$\bar{C}_{Y,j} = \cos(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j}),$$

$$\bar{S}_{X,i} = \sin(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i}),$$

and

$$\bar{S}_{Y,j} = \sin(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j}).$$

Then we can write

$$T_N = (6(\frac{1}{m} + \frac{1}{n})^{-1})^{1/2}\{\frac{1}{m}\sum_{i=1}^{m}(\ell_1\bar{C}_{X,i}+\ell_2\bar{S}_{X,i}) - \frac{1}{n}\sum_{j=1}^{n}(\ell_1\bar{C}_{Y,j}+\ell_2\bar{S}_{Y,j})\}.$$

Under H_o, $(\phi^*_{X,i}, D^*_{X,i}) \overset{d}{=} (\phi^*_{Y,j}, D^*_{Y,j})$ for all i and j, and $\phi^*_{X,i}$ is independent of $D^*_{X,i}$, since G_1 and G_2 are elliptically symmetric. Also $H_2(\phi^*_{X,i})$ and $F_2(D^*_{X,i})$ are each uniformly distribution on (0, 1), under H_o. Therefore,

$$\begin{aligned} E_{H_o}[\bar{C}_{X,i}] = E_{H_o}[\bar{C}_{Y,j}] &= E_{H_o}[\cos(2\pi H_2(\phi^*_{X,i}))F_2(D^*_{X,i})] \\ &= E_{H_o}[\cos(2\pi H_2(\phi^*_{X,i}))]E_{H_o}[F_2(D^*_{X,i})] \\ &= \int_0^1 \cos(2\pi u)du \int_0^1 udu \\ &= 0 \end{aligned}$$

and (6.1.22)

$$\begin{aligned} E_{H_o}[\bar{S}_{X,i}] = E_{H_o}[\bar{S}_{Y,j}] &= \int_0^1 \sin(2\pi u)du \int_0^1 udu \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{Var}_{H_o}[\bar{C}_{X,i}] = \mathrm{Var}_{H_o}[\bar{C}_{Y,j}] &= E_{H_o}[\bar{C}^2_{X,i}] \\ &= E_{H_o}[\cos^2(2\pi H_2(\phi^*_{X,i}))]E_{H_o}[F_2^2(D^*_{X,i})] \\ &= \int_0^1 \cos^2(2\pi u)du \int_0^1 u^2du \\ &= 1/6, \end{aligned}$$

$$\begin{aligned}\mathrm{Var}_{H_o}[\bar{S}_{X,i}] &= \mathrm{Var}_{H_o}[\bar{S}_{Y,j}] = E_{H_o}[\bar{S}^2_{X,i}] \\ &= \int_0^1 \sin^2(2\pi u)du \int_0^1 u^2 du \\ &= 1/6,\end{aligned}$$

and (6.1.23)

$$\begin{aligned}\mathrm{Cov}_{H_o}[\bar{C}_{X,i}, \bar{S}_{X,i}] &= \mathrm{Cov}_{H_o}[\bar{C}_{Y,j}, \bar{S}_{Y,j}] = E_{H_o}[\bar{C}_{X,i}\bar{S}_{X,i}] \\ &= E_{H_o}[\cos(2\pi H_2(\phi^*_{X,i}))\sin(2\pi H_2(\phi^*_{X,i}))]E_{H_o}[F_2^2(D_{X,i})] \\ &= \int_0^1 \cos(2\pi u)\sin(2\pi u)du \int_0^1 u^2 du \\ &= 0.\end{aligned}$$

Using two-sample U-statistic theory (see, e.g., Randles and Wolfe, 1979, pp. 92-93), under H_o T_N is asymptotically normal with mean 0 and variance $\ell_1^2+\ell_2^2$. It follows that, under H_o, $(6(\frac{1}{m}+\frac{1}{n})^{-1})^{1/2}(\bar{C}_{MB2}, \bar{S}_{MB2})$ is asymptotically bivariate normal with mean vector $\underline{0}$ and covariance matrix I_2. Therefore, by Lemma 6.1.18, the asymptotic null distribution of $(6(\frac{1}{m}+\frac{1}{n})^{-1})^{1/2}(C_{MB2}, S_{MB2})$ is also bivariate normal with mean vector $\underline{0}$ and covariance matrix I_2. Hence the asymptotic null distribution of

$$MB2 = ((6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{MB2})^2+((6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}S_{MB2})^2$$

is Chi-square with two degrees of freedom. □

6.2 Asymptotic Distribution of MB2 Under Contiguous Alternatives

The asymptotic distribution of MB2 under certain contiguous alternatives is easily obtained by extending the results for the

one-sample statistic, MB. We assume we are sampling from the class of densities given by 3.2.1, namely,

$$f_{\underset{\sim}{\theta}}(\underset{\sim}{y}) = c \exp\{\frac{-[(\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma_2^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})]^{\nu}}{h}\}, \text{ for } \underset{\sim}{y} \in R^2, \tag{6.2.1}$$

where $c = c(\nu)$, and $h = h(\nu)$, are given by 3.2.2, and $\nu > 0$. Again, since MB2 is invariant with respect to nonsingular linear transformations we may take $\Sigma_2 = I_2$. Since MB2 is in addition invariant with respect to translations of the observations, we may, without loss in generality, take the mean of the first population to be (0, 0) and the mean of the second population to be of the form (0, Δ). (See Section 4.1.) Le Cam's third lemma (Lemma 3.2.4) will be used to establish the following theorem.

Theorem 6.2.2. Suppose $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$ and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independent random samples from density functions $f_{\underset{\sim}{0}}$ and $f_{\underset{\sim}{\theta}_N}$, respectively, where these densities are of the form given by 6.2.1 with $\Sigma_2 = I_2$, $N = m+n$, and $\underset{\sim}{\theta}_N$ represents a sequence of alternatives:

$$H_N: \ \underset{\sim}{\theta}_N = (0, KN^{-1/2})'. \tag{6.2.3}$$

If $\frac{m}{N} \to \lambda$ as $N \to \infty$ with $0 < \lambda < 1$, then under this sequence of alternatives the asymptotic distribution of MB2 is noncentral Chi-square with two degrees of freedom and noncentrality parameter

$$\delta^2_{MB2}(\nu) = (\frac{1}{\lambda} + \frac{1}{1-\lambda})^{-1}\delta^2_{MB}(\nu).$$

Here, $\delta^2_{MB}(\nu)$, given by 3.2.30, is the noncentrality parameter of the asymptotic distribution of the one-sample statistic MB under similar contiguous alternatives.

Proof. Let p_N and q_N denote the joint density functions of $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m,\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ under H_o: $\underset{\sim}{\theta} = \underset{\sim}{0}$ and H_N: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_N = (0, KN^{-1/2})'$, respectively. That is,

$$p_N = p_N(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m, \underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n) = \prod_{i=1}^{m} f_{\underset{\sim}{0}}(\underset{\sim}{X}_i) \prod_{j=1}^{n} f_{\underset{\sim}{0}}(\underset{\sim}{Y}_j)$$

and (6.2.4)

$$q_N = q_N(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m, \underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n) = \prod_{i=1}^{m} f_{\underset{\sim}{0}}(\underset{\sim}{X}_i) \prod_{j=1}^{n} f_{\underset{\sim}{\theta}_N}(\underset{\sim}{Y}_j).$$

Denote the corresponding probability functions by P_N and Q_N. Let

$$L_N = \frac{q_N}{p_N},$$

and

$$\log L_N = \log\left[\frac{\prod_{j=1}^{n} f_{\underset{\sim}{\theta}_N}(\underset{\sim}{Y}_j)}{\prod_{j=1}^{n} f_{\underset{\sim}{0}}(\underset{\sim}{Y}_j)}\right]$$

$$= \frac{1}{h}\sum_{j=1}^{n}\{[\underset{\sim}{Y}_j'\underset{\sim}{Y}_j]^{\nu} - [(\underset{\sim}{Y}_j-\underset{\sim}{\theta}_N)'(\underset{\sim}{Y}_j-\underset{\sim}{\theta}_N)]^{\nu}\}.$$

Letting $\Delta = KN^{-1/2}$ and transforming to polar coordinates via

$$Y_{1j} = Z_j \cos(\phi^*_{Y,j})$$

and (6.2.5)

$$Y_{2j} = Z_j \sin(\phi^*_{Y,j})$$

for all $1 \le j \le n$, where $Z_j > 0$ and $0 \le \phi^*_{Y,j} < 2\pi$, we can write

$$\log L_N = \frac{1}{h} \sum_{j=1}^{n} \{Z_j^{2\nu} - (Z_j^2 - 2Z_j \sin(\phi^*_{Y,j})\Delta + \Delta^2)^{\nu}\}.$$

Again note that Z_j and $\phi^*_{Y,j}$ are independent under P_N with respective densities

$$g_1(z_j) = 2\pi c z_j e^{-\frac{z_j^{2\nu}}{h}}, \qquad 0 < z_j < \infty,$$

and (6.2.6)

$$g_2(\phi^*_{y,j}) = \frac{1}{2\pi}, \qquad 0 \le \phi^*_{y,j} < 2\pi,$$

where c, h and ν are as previously defined. The Taylor expansion used in 3.2.11 shows that

$$\log L_N = \frac{K}{\sqrt{N}h} \sum_{j=1}^{n} 2\nu \sin(\phi^*_{Y,j}) Z_j^{2\nu-1} - \frac{K^2}{Nh} \sum_{j=1}^{n} R_{Nj} \tag{6.2.7}$$

where

$$R_{Nj} = 2\nu(\nu-1)(Z_j^2 - 2Z_j \sin(\phi^*_{Y,j})\Delta_j^* + \Delta_j^{*2})^{\nu-2}(\Delta_j^{*2} - Z_j \sin(\phi^*_{Y,j}))^2$$

$$+ \nu(Z_j^2 - 2Z_j \sin(\phi^*_{Y,j})\Delta_j^* + \Delta_j^{*2})^{\nu-1},$$

for some Δ_j^* between 0 and Δ, $j = 1,\ldots,n$. In the proof of Lemma 3.2.8 we show that

$$\frac{K^2}{nh} \sum_{i=1}^{n} R_{ni} \xrightarrow{P_n} \frac{K^2 W}{h},$$

where W and R_{ni} are given in the statement of that lemma. Noting that (ϕ_i', Z_i) of Lemma 3.2.8 is equal in distribution to $(\phi_{Y,j}^*, Z_j)$ under P_N, for all i and j, here it can be seen that

$$\frac{K^2}{nh} \sum_{j=1}^{n} R_{Nj} \xrightarrow{P_N} \frac{K^2 W}{h},$$

and hence

$$\frac{K^2}{Nh} \sum_{j=1}^{n} R_{Nj} = \frac{n}{N} \frac{K^2}{nh} \sum_{j=1}^{n} R_{Nj} \xrightarrow{P_N} (1-\lambda)\frac{K^2 W}{h}.$$

Therefore

$$\log L_N = \frac{K}{\sqrt{Nh}} \sum_{j=1}^{n} 2\nu \sin(\phi_{Y,j}^*) Z_j^{2\nu-1} - \frac{(1-\lambda)K^2 W}{h} + o_{P_N}(1). \qquad (6.2.8)$$

Using the results of Lemma 3.2.23, it follows that, under P_N, $\log L_N$ is asymptotically normal with mean $(1-\lambda)\mu$ and variance $-2(1-\lambda)\mu$, where $\mu = -K^2 W h^{-1}$. So Q_N is contiguous to P_N.

If we verify that

$$T_N = \ell_1(6(\tfrac{1}{m} + \tfrac{1}{n})^{-1})^{1/2}C_{MB2} + \ell_2(6(\tfrac{1}{m} + \tfrac{1}{n})^{-1})^{1/2}S_{MB2} \qquad (6.2.9)$$

and $\log L_N$ are jointly asymptotically normal under P_N, for $(\ell_1, \ell_2) \neq (0, 0)$, then it will follow by Le Cam's third lemma that T_N is asymptotically normal under Q_N and hence $(6(\frac{1}{m} + \frac{1}{n})^{-1})^{1/2}(C_{MB2}, S_{MB2})$ will be asymptotically bivariate normal under Q_N. We, therefore, show that

$$S_N^* = ((6(\tfrac{1}{m} + \tfrac{1}{n})^{-1})^{1/2}C_{MB2}, (6(\tfrac{1}{m} + \tfrac{1}{n})^{-1})^{1/2}S_{MB2}, \log L_N) \qquad (6.2.10)$$

is asymptotically multivariate normal under P_N. Note that

$$(\tfrac{N}{m} + \tfrac{N}{n}) \to (\tfrac{1}{\lambda} + \tfrac{1}{1-\lambda}), \text{ as } N \to \infty$$

and (6.2.11)

$$\log L_N = \frac{1}{\sqrt{N}} \sum_{j=1}^{n} 2\nu K h^{-1} \sin(\phi^*_{Y,j}) Z_j^{2\nu-1} - (1-\lambda)K^2 W h^{-1} + o_{P_N}(1)$$

$$= \left(\frac{\sqrt{N}\,(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2} n}{(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2}\sqrt{N}\,\sqrt{N}}\right) \frac{1}{n} \sum_{j=1}^{n} 2\nu K h^{-1} \sin(\phi^*_{Y,j}) Z_j^{2\nu-1}$$

$$-(1-\lambda)K^2 W h^{-1} + o_{P_N}(1)$$

$$= \frac{\sqrt{N}}{(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2}} \left[(\tfrac{1-\lambda}{\lambda})^{1/2} \frac{1}{n} \sum_{j=1}^{n} 2\nu K h^{-1} \sin(\phi^*_{Y,j}) Z_j^{2\nu-1}\right]$$

$$-(1-\lambda)K^2 W h^{-1} + o_{P_N}(1).$$

Using 6.2.11 and Lemma 6.1.18, the asymptotic distribution under P_N of S_N^* is the same as that of

$$S_N = \frac{N^{1/2}}{(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2}} (\sqrt{6}\ \bar{C}_{MB2}, \sqrt{6}\ \bar{S}_{MB2}, \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ \bar{L}) - (0, 0, (1-\lambda)K^2 w n^{-1}), \tag{6.2.12}$$

where

$$\bar{L} = \frac{1}{n} \sum_{j=1}^{n} L_j,$$

$$L_j = 2\nu K h^{-1} \sin(\phi^*_{Y,j}) Z_j^{2\nu-1}, \qquad j = 1,\ldots,n,$$

and $\bar{C}_{MB2}$ and $\bar{S}_{MB2}$ are given in 6.1.7. Consider then the linear combination

$$V_N = \frac{N^{1/2}}{(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2}} (\ell_1\sqrt{6}\ \bar{C}_{MB2} + \ell_2\sqrt{6}\ \bar{S}_{MB2} + \ell_3 \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ \bar{L})$$

$$= \frac{N^{1/2}}{(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2}} (\frac{1}{m} \sum_{i=1}^{m} (\ell_1\sqrt{6}\ \bar{C}_{X,i} + \ell_2\sqrt{6}\ \bar{S}_{X,i}) - \frac{1}{n} \sum_{j=1}^{n} (\ell_1\sqrt{6}\ \bar{C}_{Y,j} + \ell_2\sqrt{6}\ \bar{S}_{Y,j} - \ell_3 \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ L_j), \tag{6.2.13}$$

where $\bar{C}_{X,i}$, $\bar{S}_{X,i}$, $\bar{C}_{Y,j}$, and $\bar{S}_{Y,j}$ are given in 6.1.21. Let

$$\sigma_C^2 = \mathrm{Var}_{P_N}[\sqrt{6}\ \bar{C}_{X,i}] = \mathrm{Var}_{P_N}[\sqrt{6}\ \bar{C}_{Y,j}],$$

$$\sigma_S^2 = \mathrm{Var}_{P_N}[\sqrt{6}\ \bar{S}_{X,i}] = \mathrm{Var}_{P_N}[\sqrt{6}\ \bar{S}_{Y,j}],$$

$$\sigma_{CS} = \mathrm{Cov}_{P_N}[\bar{C}_{X,i}, \bar{S}_{X,i}] = \mathrm{Cov}_{P_N}[\bar{C}_{Y,j}, \bar{S}_{Y,j}],$$

$$\sigma_{CL} = \mathrm{Cov}_{P_N}[\bar{C}_{Y,j}, L_j],\ \sigma_{SL} = \mathrm{Cov}_{P_N}[\bar{S}_{Y,j}, L_j],$$

$$\sigma_1^2 = \mathrm{Var}_{P_N}[\ell_1\sqrt{6}\ \bar{C}_{X,i} + \ell_2\sqrt{6}\ \bar{S}_{X,i}]$$

$$\sigma_2^2 = \mathrm{Var}_{P_N}[\ell_1\sqrt{6}\ \bar{C}_{Y,j} + \ell_2\sqrt{6}\ \bar{S}_{Y,j} - \ell_3 \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ L_j]$$

and (6.2.14)

$$\sigma_L^2 = \mathrm{Var}_{P_N}[L_j].$$

It follows, using 6.1.23, that

$$\sigma_1^2 = \ell_1^2 + \ell_2^2$$

and (6.2.15)

$$\sigma_2^2 = \ell_1^2 + \ell_2^2 + \ell_3^2(1-\lambda)\lambda^{-1}\sigma_L^2 - 2\ell_1\ell_3\sqrt{6}\ \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ \sigma_{CL} - 2\ell_2\ell_3\sqrt{6}\ \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ \sigma_{SL}.$$

Also, by 6.1.22 and 3.2.26,

$$E_{P_N}[\ell_1\sqrt{6}\ \bar{C}_{X,i} + \ell_2\sqrt{6}\ \bar{S}_{X,i}] = 0$$

and (6.2.16)

$$E_{P_N}[\ell_1\sqrt{6}\ \bar{C}_{Y,j} + \ell_2\sqrt{6}\ \bar{S}_{Y,j} - \ell_3 \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ L_j] = 0.$$

Hence the asymptotic distribution under P_N of

$$\frac{(\frac{1}{\lambda}+\frac{1}{1-\lambda})^{1/2}}{\left(\frac{\sigma_1^2}{\lambda}+\frac{\sigma_2^2}{1-\lambda}\right)^{1/2}} V_N = \frac{V_N}{((1-\lambda)\sigma_1^2+\lambda\sigma_2^2)^{1/2}} \tag{6.2.17}$$

is standard normal, as seen by applying two-sample U-statistic theory to the representation of V_N given in 6.2.13. (See, e.g., Randles and Wolfe, 1979, pp. 92-93.) Let

$$G(t) = P_{P_N}[Z_j \leq t]$$

and note (from 6.1.1) that under P_N

$$D^*_{Y,j} = \underset{\sim}{Y}'_j\underset{\sim}{Y}_j = Z_j^2, \qquad j = 1,\ldots,n \tag{6.2.18}$$

and

$$F_2(t^2) = P_{P_N}(D^*_{Y,j} \leq t^2) = P_{P_N}(Z_j \leq t) = G(t),$$

for $t \geq 0$. Using the independence of $\phi^*_{Y,j}$ and Z_j under P_N, it follows that

$$\begin{aligned}\sigma_{CL} &= \mathrm{Cov}_{P_N}[\bar{C}_{Y,j}, L_j] \\ &= 2\nu Kh^{-1}E_{P_N}[\cos(2\pi H_2(\phi^*_{Y,j}))F_2(D^*_{Y,j})\sin(\phi^*_{Y,j})Z_j^{2\nu-1}] \\ &= 2\nu Kh^{-1}E_{P_N}[\cos(2\pi H_2(\phi^*_{Y,j}))\sin(\phi^*_{Y,j})]E_{P_N}[G(Z_j)Z_j^{2\nu-1}].\end{aligned} \tag{6.2.19}$$

Since, using 6.2.6,

$$H_2(t) = P_{P_N}(\phi^*_{Y,j} \leq t) = \frac{t}{2\pi}, \tag{6.2.20}$$

we find

$$E_{P_N}[\cos(2\pi H_2(\phi^*_{Y,J}))\sin(\phi^*_{Y,J})] = E_{P_N}[\cos(\phi^*_{Y,J})\sin(\phi^*_{Y,J})]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(\phi)\sin(\phi)d\phi$$

$$= 0, \qquad (6.2.21)$$

implying $\sigma_{CL} = 0$. So

$$(1-\lambda)\sigma_1^2+\lambda\sigma_2^2 = (1-\lambda)(\ell_1^2+\ell_2^2)+\lambda(\ell_1^2+\ell_2^2)+(1-\lambda)\ell_3^2\sigma_L^2-2\ell_2\ell_3\sqrt{6}\sqrt{\lambda(1-\lambda)}\sigma_{SL}$$

$$= \ell_1^2+\ell_2^2+\ell_3^2(1-\lambda)\sigma_L^2-2\ell_2\ell_3\sqrt{6}\sqrt{\lambda(1-\lambda)}\ \sigma_{SL},$$

and (6.2.22)

$$\frac{V_N}{((1-\lambda)\sigma_1^2+\lambda\sigma_2^2)^{1/2}} = \frac{N^{1/2}}{(\frac{1}{\lambda}+\frac{1}{1-\lambda})^{1/2}}\left\{\frac{\ell_1\sqrt{6}\ \bar{C}_{MB2}+\ell_2\sqrt{6}\ \bar{S}_{MB2}+\ell_3\ \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ \bar{L}}{(\ell_1^2+\ell_2^2+\ell_3^2(1-\lambda)\sigma_L^2-2\ell_2\ell_3\sqrt{6}\sqrt{\lambda(1-\lambda)}\sigma_{SL})^{1/2}}\right\}$$

is asymptotically standard normal under P_N. Hence, under P_N

$$\frac{N^{1/2}}{(\frac{1}{\lambda}+\frac{1}{1-\lambda})^{1/2}}\ (\sqrt{6}\ \bar{C}_{MB2},\ \sqrt{6}\ \bar{S}_{MB2},\ \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\ \bar{L})$$

is asymptotically multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix

$$\Gamma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\sqrt{6}\ \sqrt{\lambda(1-\lambda)}\ \sigma_{SL} \\ 0 & -\sqrt{6}\ \sqrt{\lambda(1-\lambda)}\ \sigma_{SL} & (1-\lambda)\sigma_L^2 \end{pmatrix}.$$

The asymptotic distribution under P_N of

$$S_N^* = ((6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{MB2},\ (6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}S_{MB2},\ \log L_N)$$

is therefore also multivariate normal, with mean vector $(0,\ 0,\ -(1-\lambda)K^2Wh^{-1})$ and covariance matrix Γ_2. Thus,

$$(\ell_1(6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{MB2}+\ell_2(6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}S_{MB2},\ \log L_N)$$

is under P_N asymptotically bivariate normal with mean vector $(0,\ -(1-\lambda)K^2Wh^{-1})$ and covariance matrix

$$\Gamma_2^* = \begin{pmatrix} \ell_1^2+\ell_2^2 & -\ell_2\sqrt{6}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SL} \\ -\ell_2\sqrt{6}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SL} & (1-\lambda)\sigma_L^2 \end{pmatrix}.$$

Via Le Cam's third lemma, we conclude that, under Q_N, the asymptotic distribution of

$$\ell_1(6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{MB2}+\ell_2(6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}S_{MB2}$$

is normal with mean $=\ell_2\sqrt{6}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SL}$ and variance $\ell_1^2+\ell_2^2$. Hence, under Q_N

$$(6(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}(C_{MB2},\ S_{MB2})$$

is asymptotically bivariate normal with mean vector $(0,\ -\sqrt{6}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SL})$ and covariance matrix I_2. Finally, it follows that under Q_N

$$MB2 = 6(\tfrac{1}{m}+\tfrac{1}{n})^{-1}(C_{MB2}^2+S_{MB2}^2)$$

is asymptotically Chi-square with two degrees of freedom and noncentrality parameter,

$$\delta^2_{MB2}(\nu) = 6(1-\lambda)\lambda\sigma^2_{SL}. \tag{6.2.23}$$

Referring to 6.2.18, we find

$$\begin{aligned}\sigma_{SL} &= \mathrm{Cov}_{P_N}[\bar{S}_{Y,j}, L_j] \\ &= E_{P_N}[\bar{S}_{Y,j}, L_j] \\ &= 2\nu Kh^{-1}E_{P_N}[\sin(2\pi H_2(\phi^*_{Y,j}))\sin(\phi^*_{Y,j})]E_{P_N}[G(Z_j)Z_j^{2\nu-1}].\end{aligned}$$

Since, using 6.2.6 and 6.2.20,

$$\begin{aligned}E_{P_N}[\sin(2\pi H_2(\phi^*_{Y,j}))\sin(\phi^*_{Y,j})] &= E_{P_N}[\sin^2(\phi^*_{Y,j})] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sin^2(\phi^*)d\phi^* \\ &= 1/2,\end{aligned} \tag{6.2.24}$$

it follows via 3.2.39 and 3.2.40 that $\sigma_{SL} = \mathrm{Cov}_{P_N}[\bar{S}_1, L_1]$, as given in 3.2.42. Hence

$$\delta^2_{MB2}(\nu) = \lambda(1-\lambda)\delta^2_{MB}(\nu),$$

where $\delta^2_{MB}(\nu)$ is given in 3.2.43. □

6.3 Asymptotic Distributions of Mardia's Statistic and a Modification

Mardia (1967) shows heuristically that his statistic, as defined in 5.1.3 and 5.1.4, is under the null hypothesis asymptotically Chi-square with two degrees of freedom. A heuristic derivation of the asymptotic distribution under alternatives is also given by Mardia. In this section, we will find the asymptotic distributions, under the

null hypothesis and under contiguous alternatives, of the modification of Mardia's statistic defined in 5.1.8. With H_N as defined in 6.1, write

$$C_{M2} = \frac{1}{m}\sum_{i=1}^{m} \cos(2\pi H_N(\phi_{X,i})) - \frac{1}{n}\sum_{j=1}^{n} \cos(2\pi H_N(\phi_{Y,j}))$$

$$S_{M2} = \frac{1}{m}\sum_{i=1}^{m} \sin(2\pi H_N(\phi_{X,i})) - \frac{1}{n}\sum_{j=1}^{n} \sin(2\pi H_N(\phi_{Y,j})), \qquad (6.3.1)$$

and recall that the modification of Mardia's statistic that we will use is

$$M2 = 2(\frac{1}{m} + \frac{1}{n})^{-1}\{C_{M2}^2 + S_{M2}^2\}.$$

Using the definitions of $\phi^*_{X,i}$, $\phi^*_{Y,j}$, and H_2 as given in section 1 of this chapter, we introduce a statistic asymptotically equivalent to M2 under the null hypothesis. Let

$$\bar{C}_{M2} = \frac{1}{m}\sum_{i=1}^{m} \cos(2\pi H_2(\phi^*_{X,i})) - \frac{1}{n}\sum_{j=1}^{n} \cos(2\pi H_2(\phi^*_{Y,j}))$$

and

$$\bar{S}_{M2} = \frac{1}{m}\sum_{i=1}^{m} \sin(2\pi H_2(\phi^*_{X,i})) - \frac{1}{n}\sum_{j=1}^{n} \sin(2\pi H_2(\phi^*_{Y,j})). \qquad (6.3.2)$$

The following lemma shows that $(\frac{1}{m} + \frac{1}{n})^{-1/2}(C_{M2}, S_{M2})$ and $(\frac{1}{m} + \frac{1}{n})^{-1/2}(\bar{C}_{M2}, \bar{S}_{M2})$ have the same asymptotic null distribution.

Lemma 6.3.3. Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$ and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be independent random samples from bivariate populations with continuous distribution functions $G_1(u_1, u_2)$ and $G_2(u_1, u_2) = G_1(u_1-\theta_1, u_2-\theta_2)$, respectively. If under H_o: $\underset{\sim}{\theta} = (\theta_1, \theta_2)' = \underset{\sim}{0}$, $E[\underset{\sim}{X}_i] = E[\underset{\sim}{Y}_j] = \underset{\sim}{\mu}$ is finite, $\mathrm{Var}[\underset{\sim}{X}_i] = \mathrm{Var}[\underset{\sim}{Y}_i] = \Sigma_2$ is positive definite, and $\lim_{N\to\infty} \frac{m}{N} = \lambda$, $0 < \lambda < 1$, then

$$(\tfrac{1}{m} + \tfrac{1}{n})^{-1/2}(C_{M2}-\bar{C}_{M2},\ S_{M2}-\bar{S}_{M2}) \overset{H_o}{\to} (0,\ 0),$$

as $N \to \infty$.

Proof. Referring to the proof of Lemma 6.1.18, we see that we can write

$$E_{H_o}[\{(\tfrac{1}{m} + \tfrac{1}{n})^{-1/2}(C_{M2}-\bar{C}_{M2})\}^2] = E_{H_o}[U_i^{*2}]-E_{H_o}[U_i^*U_k^*], \quad i \neq k \tag{6.3.4}$$

where

$$U_i^* = \cos(2\pi H_N(\phi_{X,i}))-\cos(2\pi H_2(\phi^*_{X,i})),$$

for $i = 1,\ldots,m$. An examination of the proof of Lemma 6.1.16 shows that the conditions stated here are sufficient for 6.1.12 and 6.1.15. These two conditions were shown to imply, via Lemma 3.1.4, that

$$|H_N(\phi_{X,i})-H_2(\phi^*_{X,i})| \overset{H_o}{=} o_p(1). \tag{6.3.5}$$

Hence, $U_i^* \overset{H_o}{=} o_p(1)$ and, by Lebesgue Dominated Convergence Theorem,

$$E_{H_o}[U_1^{*2}] - E_{H_o}[U_1^* U_k^*] = o(1).$$

Therefore,

$$(\frac{1}{m} + \frac{1}{n})^{-1/2}(C_{M2} - \bar{C}_{M2}) \overset{H_o}{=} o_p(1)$$

and, similarly,

$$(\frac{1}{m} + \frac{1}{n})^{-1/2}(S_{M2} - \bar{S}_{M2}) \overset{H_o}{=} o_p(1). \quad \square$$

Using Lemma 6.3.3 we establish the following theorem.

Theorem 6.3.6. Under the conditions of Lemma 6.3.3, the asymptotic null distribution of M2 is Chi-square with two degrees of freedom.

Proof. Let

$$T_N^* = (2(\frac{1}{m} + \frac{1}{n})^{-1})^{1/2}\{\ell_1 \bar{C}_{M2} + \ell_2 \bar{S}_{M2}\}$$

$$= (2(\frac{1}{m} + \frac{1}{n})^{-1})^{1/2}\{\frac{1}{m}\sum_{i=1}^{m}(\ell_1 \bar{C}_{MX,i} + \ell_2 \bar{S}_{MX,i})$$

$$- \frac{1}{n}\sum_{j=1}^{n}(\ell_1 \bar{C}_{MY,j} + \ell_2 \bar{S}_{MY,j})\},$$

where, for $1 \le i \le m$ and $1 \le j \le n$, (6.3.7)

$$\bar{C}_{MX,i} = \cos(2\pi H_2(\phi_{X,i}^*)),$$

$$\bar{C}_{MY,j} = \cos(2\pi H_2(\phi^*_{Y,j})),$$

$$\bar{S}_{MX,i} = \sin(2\pi H_2(\phi^*_{X,i})),$$

and

$$\bar{S}_{MY,j} = \sin(2\pi H_2(\phi^*_{Y,j})).$$

Since $H_2(\phi^*_{X,i})$ and $H_2(\phi^*_{Y,j})$ are each uniformly distributed on (0,1) under the null hypothesis it follows that, for all i and j,

$$E_{H_o}[\bar{C}_{MX,i}] = E_{H_o}[\bar{C}_{MY,j}] = E_{H_o}[\bar{S}_{MX,i}] = E_{H_o}[\bar{S}_{MY,j}] = 0,$$

$$Var_{H_o}[\bar{C}_{MX,i}] = Var_{H_o}[\bar{C}_{MY,j}] = Var_{H_o}[\bar{S}_{MX,i}] = Var_{H_o}[\bar{S}_{MY,j}] = \frac{1}{2},$$

and (6.3.8)

$$Cov_{H_o}[\bar{C}_{MX,i}, \bar{S}_{MX,i}] = Cov_{H_o}[\bar{C}_{MY,j}, \bar{S}_{MY,j}] = 0.$$

Hence, by U-statistic theory, T^*_N is under H_o asymptotically normal with mean 0 and variance $\ell_1^2+\ell_2^2$. The asymptotic null distribution of $(2(\frac{1}{m}+\frac{1}{n})^{-1})^{1/2}(\bar{C}_{M2}, \bar{S}_{M2})$ is, thus, bivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix I_2. Lemma 6.3.3 shows that this is also the asymptotic null distribution of $(2(\frac{1}{m}+\frac{1}{n})^{-1})^{1/2}(C_{M2}, S_{M2})$. The asymptotic null distribution of M2 follows. □

The asymptotic distributions under continguous alternatives of M2 and B, Blumen's one-sample statistic, are related in the same way as that of the two-sample statistic, MB2, and the one-sample statistic, MB. We next obtain the asymptotic distribution of M2 under contiguous alternatives.

Theorem 6.3.9. Under the conditions of Theorem 6.2.2, the asymptotic distribution under H_N: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_N = (0, KN^{-1/2})$ of M2 is noncentral Chi-square with two degrees of freedom and noncentrality parameter

$$\delta^2_{M2}(\nu) = (\frac{1}{\lambda} + \frac{1}{1-\lambda})^{-1}\delta^2_B(\nu),$$

where $\delta^2_B(\nu)$, given by 3.3.7, is the noncentrality parameter of Blumen's one-sample statistic under Pitman alternatives.

Proof. We proceed exactly as in the proof of Theorem 6.2.2. Replace V_N in 6.2.13 by

$$V_N^* = \frac{N^{1/2}}{(\frac{1}{\lambda} + \frac{1}{1-\lambda})^{1/2}} \{\frac{1}{m}\sum_{i=1}^{m} (\ell_1\sqrt{2}\ \bar{C}_{MX,i} + \ell_2\sqrt{2}\ \bar{S}_{MX,i}) - \frac{1}{n}\sum_{j=1}^{n} (\ell_1\sqrt{2}\ \bar{C}_{MY,j} + \ell_2\sqrt{2}\ \bar{S}_{MY,j} - \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}} L_j)\} \tag{6.3.10}$$

Let L_j and $\bar{L}$ be as defined in 6.2.12 and define

$$\sigma_{CM,L} = \mathrm{Cov}_{P_N}[\bar{C}_{MY,j},\ L_j],$$

$$\sigma_{SM,L} = \mathrm{Cov}_{P_N}[\bar{S}_{MY,j},\ L_j],$$

$$\sigma_1'^2 = \mathrm{Var}_{P_N}[\ell_1\sqrt{2}\ \bar{C}_{MX,i} + \ell_2\sqrt{2}\ \bar{S}_{MX,i}],$$

and

$$\sigma_2'^2 = \mathrm{Var}_{P_N}[\ell_1\sqrt{2}\ \bar{C}_{MY,j} + \ell_2\sqrt{2}\ \bar{S}_{MY,j} - \ell_3\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}} L_j].$$

By 6.3.8 $\sigma_1'^2 = \ell_1^2+\ell_2^2$ and write

$$\sigma_2'^2 = \ell_1^2+\ell_2^2+\ell_3^2((1-\lambda)\lambda^{-1})\sigma_L^2$$
$$-2\ell_1\ell_3\sqrt{2}\,\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\,\sigma_{CM,L}$$
$$-2\ell_2\ell_3\sqrt{2}\,\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\,\sigma_{SM,L}.$$

Appealing to 6.2.19-6.2.21, it is seen that $\sigma_{CM,L} = 0$. Following the approach given in the proof of Theorem 6.2.2 (see 6.2.16 and 6.2.17) we find that

$$\frac{V_N^*}{((1-\lambda)\sigma_1'^2+\lambda\sigma_2'^2)} = \frac{N^{1/2}}{(\frac{1}{\lambda}+\frac{1}{1-\lambda})^{1/2}}\left\{\frac{\ell_1\sqrt{2}\,\bar{C}_{M2}+\ell_2\sqrt{2}\,\bar{S}_{M2}+\ell_3\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\bar{L}}{(\ell_1^2+\ell_2^2+\ell_3^2(1-\lambda)\sigma_L^2-2\ell_2\ell_3\sqrt{2}\sqrt{(1-\lambda)\lambda}\sigma_{SM,L})^{1/2}}\right\}$$

is asymptotically standard normal under P_N. This implies that

$$\frac{N^{1/2}}{(\frac{1}{\lambda}+\frac{1}{1-\lambda})^{1/2}}\,(\sqrt{2}\,\bar{C}_{M2},\ \sqrt{2}\,\bar{S}_{M2},\ \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\,\bar{L})$$

is, under P_N, asymptotically trivariate normal with mean vector $\underset{\sim}{0}$, and covariance matrix

$$\Gamma_{M2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\sqrt{2}\,\sqrt{(1-\lambda)\lambda}\,\sigma_{SM,L} \\ 1 & -\sqrt{2}\,\sqrt{(1-\lambda)\lambda}\,\sigma_{SM,L} & (1-\lambda)\,\sigma_L^2 \end{pmatrix}.$$

The asymptotic distribution under P_N of

$$((2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{M2},\ (2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})S_{M2},\ \log L_N)$$

is therefore, using Lemma 6.3.3 and the proof of Theorem 6.2.2, also trivariate normal with mean vector $(0,\ 0,\ -(1-\lambda)K^2wh^{-1})$, and covariance matrix Γ_2. Under P_N,

$$(\ell_1(2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{M2}+\ell_2(2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}S_{M2},\ \log L_N)$$

is asymptotically bivariate normal with mean vector, $(0,\ -(1-\lambda)K^2wh^{-1})$, and covariance matrix,

$$\begin{pmatrix} \ell_1^2+\ell_2^2 & -\ell_2\sqrt{2}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SM,L} \\ -\ell_2\sqrt{2}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SM,L} & (1-\lambda)\sigma_L^2 \end{pmatrix}.$$

Le Cam's third lemma implies that, under the sequence of alternatives Q_N, the asymptotic distribution of

$$\ell_1(2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}C_{M2}+\ell_2(2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}S_{M2}$$

is normal with mean $-\ell_2\sqrt{2}\ \sqrt{(1-\lambda)\lambda}\ \sigma_{SM,L}$ and variance $\ell_1^2+\ell_2^2$. Therefore

$$(2(\tfrac{1}{m}+\tfrac{1}{n})^{-1})^{1/2}(C_{M2},\ S_{M2})$$

is asymptotically bivariate normal under Q_N with mean vector,

$(0, -\sqrt{2}\sqrt{(1-\lambda)\lambda}\,\sigma_{SM,L})$, and covariance matrix, I_2. Under Q_N, the asymptotic distribution of M2 is thus noncentral Chi-square with two degrees of freedom and noncentrality parameter

$$\delta^2_{M2}(\nu) = 2(1-\lambda)\lambda\sigma^2_{SM,L}.$$

Furthermore

$$\sigma_{SM,L} = \mathrm{Cov}_{P_N}[\bar{S}_{MY,j}, L_j]$$

$$= 2\nu Kh^{-1}E_{P_N}[\sin(2\pi H_2(\phi^*_{Y,j}))\sin(\phi^*_{Y,j})Z_j^{2\nu-1}]$$

$$= \mathrm{Cov}_{P_N}[\bar{S}_{Bi}, L_i]$$

$$= \left(\frac{\delta^2_B(\nu)}{2}\right)^{1/2},$$

using 3.3.11, 3.3.12, 3.2.40, and 6.2.24. □

It can be checked that the heuristically developed expression for the noncentrality parameter of Mardia's statistic, given by 12.19 of his 1967 paper, agrees with $\delta^2_{M2}(\nu)$ for the contiguous alternatives under consideration.

CHAPTER 7
COMPARISONS OF TWO-SAMPLE TEST STATISTICS

7.1 Asymptotic Relative Efficiencies

In this section we develop asymptotic comparisons of the proposed statistic, MB2, with the modification of Mardia's statistic, M2, and with the two-sample Hotelling's T_2^2 (see 1.1.11). Again due to the affine-invariance of these statistics, we may, without loss of generality, assume that the common population covariance matrix is the identity and we need only consider alternatives of the form

H_N: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_N = (0, \Delta)'$.

Under H_N, with $\Delta = KN^{-1/2}$, if $\frac{m}{N} \to \lambda$, $0 < \lambda < 1$, as $N \to \infty$, then the asymptotic distribution of T_2^2 is Chi-square with two degrees of freedom and noncentrality parameter

$$\delta^2_{T_2^2} = (\frac{1}{\lambda} + \frac{1}{1-\lambda})^{-1}K^2. \tag{7.1.1}$$

(See, e.g., Puri and Sen, 1971, p. 212.) This result in conjunction with Hannan's work (see 4.1.2) and Theorems 6.2.2 and 6.3.9 verifies the following result.

Theorem 7.1.2. Suppose $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_m$ and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ are independent random samples from density functions $f_{\underset{\sim}{0}}$ and $f_{\underset{\sim}{\theta}_N}$, respectively, where these densities are of the form given by 6.2.1 with $\Sigma_2 = I_2$ and $N = m+n$. For testing

$$H_o: \underset{\sim}{\theta} = \underset{\sim}{0} \quad \text{against} \quad H_a: \underset{\sim}{\theta} = \underset{\sim}{\theta}_N,$$

if $\nu > 1/2$ and $\lim_{N\to\infty} \frac{m}{N} = \lambda$, $0 < \lambda < 1$, then

$$ARE(M2, T_2^2) = \frac{\delta^2_{M2}(\nu)}{\delta^2_{T_2^2}} = \frac{\delta^2_B(\nu)}{\delta^2_T},$$

and

$$ARE(MB2,\ T_2^2) = \frac{\delta^2_{MB2}(\nu)}{\delta^2_{T_2^2}} = \frac{\delta^2_{MB}(\nu)}{\delta^2_T}.$$

That is, the Pitman asymptotic relative efficiencies of MB2 and M2 relative to T_2^2 are precisely those of the one-sample versions, MB and B, respectively, relative to the one-sample Hotelling's T^2. Table 4.1 gave one-sample Pitman asymptotic relative efficiencies for selected values of ν. For convenience, we display these efficiencies again in Table 7.1.

Table 7.1
Pitman Asyptotic Relative Efficiencies

ν	$ARE(M2,\ T_2^2)$	$ARE(MB2,\ T_2^2)$
10.0	.50530	1.33231
5.0	.51872	1.22101
1.0	.78540	.98464
.8	.91839	.99574
.7	1.02988	1.01488
.5$^+$	1.50000	1.12500

Corresponding to the results in the one-sample setting, MB2 is quite competitive with T_2^2 for all the distributions considered. It dominates M2 and T_2^2 for light-tailed distributions. M2, on the other hand, performs best asymptotically for heavy-tailed distributions. The conjectured efficiencies from Table 4.2 for $0 < \nu < 1/2$ are given in Table 7.2.

Table 7.2
Conjectured Pitman Asymptotic
Relative Efficiencies

ν	ARE(M2, T_2^2)	ARE(MB2, T_2^2)
.4	2.09838	1.27541
.3	3.69648	1.64385
.2	11.59689	3.00306
.1	366.57838	25.03483

7.2 Monte Carlo Results

In this section we compare several two-sample procedures using Monte Carlo simulations.[1] The affine invariant statistics M, T_2^2, and MB2 are examined along with the bivariate Wilcoxon Rank Sum statistic proposed by Chatterjee and Sen (1964). As discussed in Chapter 1, the statistic of Chatterjee and Sen is not affine invariant. It is of the form:

[1] All calculations and random variable generations were performed with an IBM-4341 running on a vm/cms operating system. Fortran 77 and several IMSL library subroutines were used.

$$[\tfrac{1}{N+1}(S_1,\ S_2)]'(NV)^{-1}[\tfrac{1}{N+1}(S_1,\ S_2)],$$

where S_1 and S_2 are the coordinate-wise Wilcoxon Rank Sum statistics, as given in 1.1.12, and V is a consistent estimator of the asymptotic covariance matrix of $\frac{1}{\sqrt{N}(N+1)}(S_1,\ S_2)$. Using Hettmansperger (1984, pp. 289-290), we chose $\frac{mn}{12(N)(N+1)}$ for the diagonal elements of V and

$$\frac{mn}{N^2(N-1)}\left\{\sum_{i=1}^{m}\frac{\text{Rank}(X_{1i})\text{Rank}(X_{2i})}{(N+1)^2}+\sum_{j=1}^{n}\frac{\text{Rank}(Y_{1i})\text{Rank}(Y_{2i})}{(N+1)^2}-\frac{1}{4}\right\}$$

for the off-diagonal elements.

The four statistics M, T_2^2, MB2, and CS were compared using samples from the class given in 6.2.1, Pearson Type II (light-tailed) and Type VII (heavy-tailed) distributions, and bivariate normal mixtures. (These same distributions were examined in comparing the corresponding one-sample statistics. See Section 4.2 of Chapter 4 for a more detailed discussion of these distributions and the generation procedure.)

The performances of M, T_2^2, and MB2 parallel the performances of their one-sample counterparts, namely, B, T^2, and MB, respectively.

Looking at the empirical powers in Table 7.3 (m = n = 15, significance level = .05813 as in the one-sample case), we see that Mardia's statistic, M, has the greatest power for the heaviest-tailed distributions (ν = .10 and ν = .50) and Hotelling's T_2^2 performs quite poorly in the heaviest-tailed situation. For the bivariate normal

Table 7.3
Two-Sample Monte Carlo Results with
m = 15, n = 15, and α = .05318

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
		ν = .10		
(.000,.000)	.057	.040 (.025)	.048	.047
(.014,.014)	.068	.044 (.025)	.055	.052
(.028,.028)	.100	.045 (.026)	.066	.075
(.042,.042)	.146	.051 (.029)	.086	.108
		ν = .50		
(.000,.000)	.052	.075 (.049)	.065	.051
(.141,.141)	.070	.102 (.070)	.083	.079
(.283,.283)	.162	.191 (.149)	.178	.161
(.424,.424)	.328	.355 (.279)	.315	.308
		ν = 1.00		
(.000,.000)	.059	.086 (.060)	.067	.063
(.141,.141)	.068	.104 (.082)	.087	.066
(.283,.283)	.118	.183 (.143)	.164	.133
(.424,.424)	.204	.334 (.276)	.293	.247
		ν = 50.00		
(.000,.000)	.049	.078 (.063)	.073	.046
(.141,.141)	.057	.105 (.073)	.090	.064
(.283,.283)	.090	.188 (.146)	.175	.129
(.424,.424)	.142	.314 (.250)	.305	.213
		ν = 120.00		
(.000,.000)	.050	.075 (.051)	.059	.042
(.141,.141)	.058	.095 (.074)	.083	.071
(.283,.283)	.082	.172 (.134)	.175	.117
(.424,.424)	.110	.289 (.238)	.337	.216

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample sizes: m = 15, n = 15.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

distribution ($\nu = 1.0$) and distributions close to the bivariate normal, MB2 and T_2^2 perform similarly. For very light-tailed distributions ($\nu = 120$), MB2 has greater power than each of the three competitors included in the study. The statistic of Chatterjee and Sen performs well, although again problems due to its nonaffine-invariant nature can be seen in Table 7.4. A dramatic loss in power occurs as the correlation decreases to -.99.

Changes in significance levels and/or sample sizes (Tables 7.5 - 7.10) produce similar results. For some of the sample sizes examined, the exact null distribution of Mardia's statistic was available. (See Mardia, 1967, and Mardia, 1969.) In these cases, critical values were included from the exact and the asymptotic null distributions. Significance levels were selected from the naturally occurring percentiles and nonrandomized tests were performed using both critical values. An inspection of the simulated powers under the null hypothesis suggests that samples of size 12 or so may be needed to use the asymptotic null distribution of MB2.

Simulations using the light-tailed Pearson Type II and heavy-tailed Pearson Type VII distributions (Table 7.11) yield the same general pattern of results. The bivariate normal mixtures studied (Table 7.12) suggest that MB2, like MB, is robust under departures from elliptical symmetry. (Recall Mixtures 1, 2, and 3 are not elliptically symmetric, with Mixture 3 being bimodal.) Lastly, the poor performance of Hotelling's T_2^2 relative to MB2 is demonstrated for heavy-tailed mixtures (Mixtures 4 and 5).

Table 7.4
Two-Sample Monte Carlo Results with m = 15, n = 15,
α = .05318, and Varying Correlation Structures

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
	Bivariate Normal (unit variances, correlation = 0.0)			
(.000,.000)	.041	.072 (.048)	.049	.051
(.141,.141)	.055	.105 (.065)	.075	.081
(.284,.284)	.117	.183 (.145)	.153	.172
(.424,.424)	.211	.325 (.256)	.280	.290
	Bivariate Normal (unit variances, correlation = -.50)			
(.000,.000)	.048	.063 (.048)	.057	.048
(.10,.10)	.061	.094 (.056)	.082	.057
(.20,.20)	.094	.167 (.128)	.141	.111
(.30,.30)	.194	.315 (.254)	.265	.213
	Bivariate Normal (unit variances, correlation = -.99)			
(.000,.000)	.055	.082 (.054)	.056	.049
(.014,.014)	.064	.103 (.077)	.080	.059
(.028,.028)	.120	.184 (.145)	.157	.084
(.042,.042)	.203	.334 (.261)	.273	.132

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample sizes: m = 15, n = 15.
Distributions: Bivariate normal.

Table 7.5
Two-Sample Monte Carlo Results with m = 15, n = 15, and α = .10

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
		ν = .10		
(.000,.000)	.100	.099 (.064)	.117	.092
(.014,.014)	.125	.101 (.068)	.122	.108
(.028,.028)	.172	.104 (.071)	.147	.144
(.042,.042)	.241	.114 (.075)	.180	.202
		ν = .50		
(.000,.000)	.100	.133 (.098)	.118	.102
(.141,.141)	.152	.168 (.138)	.167	.147
(.283,.283)	.279	.317 (.260)	.294	.277
(.424,.424)	.474	.496 (.437)	.461	.467
		ν = 1.00		
(.000,.000)	.100	.121 (.098)	.118	.101
(.141,.141)	.126	.173 (.137)	.154	.133
(.283,.283)	.211	.277 (.234)	.242	.234
(.424,.424)	.334	.451 (.400)	.410	.374
		ν = 50.00		
(.000,.000)	.103	.135 (.104)	.127	.103
(.141,.141)	.124	.166 (.139)	.161	.133
(.283,.283)	.182	.280 (.237)	.277	.220
(.424,.424)	.256	.438 (.389)	.443	.353
		ν = 120.00		
(.000,.000)	.107	.128 (.095)	.112	.099
(.141,.141)	.121	.173 (.135)	.168	.136
(.283,.283)	.150	.270 (.225)	.306	.229
(.424,.424)	.201	.425 (.367)	.500	.352

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .10.
Number of samples: 1000 from each distribution.
Sample sizes: m = 15, n = 15.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

Table 7.6
Two-Sample Monte Carlo Results with m = 15, n = 15, and α = .01

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
		ν = .10		
(.000,.000)	.009	.008 (.002)	.009	.008
(.014,.014)	.010	.009 (.003)	.011	.013
(.028,.028)	.019	.009 (.005)	.017	.015
(.042,.042)	.034	.011 (.007)	.023	.027
		ν = .50		
(.000,.000)	.010	.020 (.010)	.004	.007
(.141,.141)	.014	.029 (.011)	.017	.015
(.283,.283)	.048	.077 (.046)	.045	.043
(.424,.424)	.119	.172 (.104)	.100	.095
		ν = 1.00		
(.000,.000)	.012	.025 (.010)	.014	.010
(.141,.141)	.016	.039 (.022)	.024	.016
(.283,.283)	.033	.075 (.043)	.046	.033
(.424,.424)	.065	.171 (.099)	.094	.071
		ν = 50.00		
(.000,.000)	.013	.033 (.015)	.014	.012
(.141,.141)	.011	.048 (.020)	.020	.014
(.283,.283)	.025	.082 (.047)	.048	.032
(.424,.424)	.040	.153 (.097)	.103	.069
		ν = 120.00		
(.000,.000)	.012	.031 (.012)	.014	.010
(.141,.141)	.011	.040 (.025)	.018	.017
(.283,.283)	.014	.070 (.037)	.057	.028
(.424,.424)	.028	.148 (.094)	.135	.069

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .01.
Number of samples: 1000 from each distribution.
Sample sizes: m = 15, n = 15.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

Table 7.7
Two-Sample Monte Carlo Results with
$m = 15$, $n = 10$, and $\alpha = .05318$

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
		$\nu = .10$		
(.000,.000)	.044	.032 (.015)	.058	.035
(.014,.014)	.059	.031 (.019)	.071	.040
(.028,.028)	.092	.040 (.020)	.086	.078
(.042,.042)	.132	.053 (.022)	.116	.112
		$\nu = .50$		
(.000,.000)	.038	.061 (.036)	.059	.033
(.141,.141)	.076	.096 (.063)	.083	.061
(.283,.283)	.152	.190 (.127)	.154	.129
(.424,.424)	.289	.335 (.254)	.285	.271
		$\nu = 1.00$		
(.000,.000)	.039	.067 (.039)	.046	.029
(.141,.141)	.061	.097 (.061)	.077	.063
(.283,.283)	.092	.169 (.125)	.141	.108
(.424,.424)	.175	.310 (.225)	.235	.199
		$\nu = 50.00$		
(.000,.000)	.034	.078 (.049)	.064	.037
(.141,.141)	.052	.091 (.067)	.083	.052
(.283,.283)	.078	.163 (.113)	.136	.098
(.424,.424)	.135	.285 (.212)	.252	.178
		$\nu = 120.00$		
(.000,.000)	.037	.071 (.044)	.055	.046
(.141,.141)	.050	.093 (.065)	.080	.056
(.283,.283)	.067	.170 (.111)	.167	.106
(.424,.424)	.101	.301 (.214)	.308	.192

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample sizes: $m = 15$, $n = 10$.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

Table 7.8
Two-Sample Monte Carlo Results with
m = 12, n = 12, and α = .05318

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
		ν = .10		
(.000,.000)	.040	.042 (.019)	.051	.046
(.014,.014)	.055	.048 (.021)	.062	.053
(.028,.028)	.088	.053 (.028)	.074	.071
(.042,.042)	.120	.063 (.033)	.089	.095
		ν = .50		
(.000,.000)	.049	.072 (.039)	.067	.043
(.141,.141)	.081	.106 (.067)	.082	.067
(.283,.283)	.145	.192 (.145)	.168	.147
(.424,.424)	.277	.317 (.242)	.275	.278
		ν = 1.00		
(.000,.000)	.058	.077 (.052)	.051	.045
(.141,.141)	.059	.102 (.065)	.067	.063
(.283,.283)	.099	.181 (.129)	.150	.116
(.424,.424)	.181	.313 (.243)	.250	.227
		ν = 50.00		
(.000,.000)	.052	.077 (.051)	.061	.039
(.141,.141)	.051	.095 (.061)	.078	.051
(.283,.283)	.078	.169 (.107)	.144	.087
(.424,.424)	.119	.288 (.212)	.262	.182
		ν = 120.00		
(.000,.000)	.051	.071 (.048)	.057	.042
(.141,.141)	.058	.089 (.059)	.082	.051
(.283,.283)	.067	.157 (.103)	.156	.096
(.424,.424)	.097	.273 (.198)	.288	.174

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample sizes: m = 12, n = 12.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

Table 7.9
Two-Sample Monte Carlo Results with
m = 10, n = 10, and α = .04936

Bivariate median difference	Statistic					
	M		T_2^2		MB2	CS
			ν = .10			
(.000,.000)	.051	(.052)	.051	(.023)	.058	.037
(.014,.014)	.057	(.062)	.052	(.023)	.067	.046
(.028,.028)	.069	(.078)	.056	(.028)	.075	.061
(.042,.042)	.087	(.097)	.062	(.032)	.086	.086
			ν = .50			
(.000,.000)	.053	(.057)	.078	(.042)	.058	.042
(.141,.141)	.055	(.064)	.117	(.059)	.077	.052
(.283,.283)	.105	(.117)	.175	(.117)	.133	.109
(.424,.424)	.208	(.216)	.283	(.206)	.241	.202
			ν = 1.00			
(.000,.000)	.055	(.060)	.093	(.047)	.064	.043
(.141,.141)	.065	(.072)	.104	(.063)	.078	.048
(.283,.283)	.086	(.096)	.165	(.100)	.113	.084
(.424,.424)	.144	(.156)	.246	(.176)	.179	.154
			ν = 50.00			
(.000,.000)	.054	(.059)	.108	(.065)	.075	.049
(.141,.141)	.054	(.060)	.114	(.077)	.086	.064
(.283,.283)	.075	(.080)	.155	(.097)	.122	.087
(.424,.424)	.111	(.119)	.244	(.162)	.198	.126
			ν = 120.00			
(.000,.000)	.057	(.065)	.106	(.062)	.067	.058
(.141,.141)	.058	(.065)	.116	(.076)	.085	.064
(.283,.283)	.069	(.077)	.164	(.105)	.133	.091
(.424,.424)	.100	(.103)	.234	(.165)	.211	.149

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 and Mardia's statistic using the $F_{2,N-3}$ distribution and the exact null distribution of M, respectively.
Nominal significance level: .04936.
Number of samples: 1000 from each distribution.
Sample sizes: m = 10, n = 10.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

Table 7.10
Two-Sample Monte Carlo Results with
m = 8, n = 8, and α = .04973

Bivariate median difference	Statistic					
	M		T_2^2		MB2	CS
			ν = .10			
(.000,.000)	.051	(.044)	.052	(.011)	.065	.035
(.014,.014)	.049	(.040)	.053	(.009)	.065	.040
(.028,.028)	.072	(.064)	.055	(.013)	.075	.047
(.042,.042)	.100	(.090)	.059	(.021)	.092	.067
			ν = .50			
(.000,.000)	.047	(.043)	.076	(.042)	.063	.042
(.141,.141)	.054	(.048)	.108	(.049)	.079	.042
(.283,.283)	.092	(.084)	.158	(.086)	.109	.084
(.424,.424)	.160	(.147)	.259	(.155)	.195	.150
			ν = 1.00			
(.000,.000)	.056	(.052)	.109	(.047)	.075	.040
(.141,.141)	.063	(.058)	.111	(.065)	.083	.053
(.283,.283)	.080	(.072)	.155	(.094)	.108	.075
(.424,.424)	.123	(.108)	.234	(.146)	.166	.125
			ν = 50.00			
(.000,.000)	.045	(.041)	.098	(.057)	.063	.046
(.141,.141)	.047	(.042)	.114	(.062)	.069	.055
(.283,.283)	.061	(.056)	.148	(.082)	.102	.065
(.424,.424)	.095	(.083)	.227	(.133)	.163	.100
			ν = 120.00			
(.000,.000)	.045	(.039)	.085	(.052)	.050	.036
(.141,.141)	.050	(.042)	.091	(.059)	.060	.044
(.283,.283)	.059	(.052)	.133	(.075)	.087	.057
(.424,.424)	.073	(.068)	.203	(.119)	.158	.096

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 and Mardia's statistic using the $F_{2,N-3}$ distribution and the exact null distribution of M, respectively.
Nominal significance level: .04973.
Number of samples: 1000 from each distribution.
Sample sizes: m = 8, n = 8.
Distributions: Class given in 6.2.1, $\Sigma_2 = I_2$.

Table 7.11
Two-Sample Monte Carlo Results for Pearson Type II and Type VII Distributions with m = 15, n = 15, and α = .05318

Bivariate median difference	Statistic			
	M	T_2^2	MB2	CS
		Bivariate Normal (I = I)		
(.000,.000)	.058	.088 (.063)	.068	.057
(.014,.014)	.072	.113 (.083)	.086	.085
(.028,.028)	.122	.190 (.146)	.152	.128
(.042,.042)	.224	.350 (.277)	.291	.255
		Pearson Type II (m = 1)		
(.000,.000)	.055	.091 (.058)	.065	.058
(.141,.141)	.062	.111 (.086)	.088	.078
(.283,.283)	.096	.180 (.143)	.170	.127
(.424,.424)	.151	.340 (.269)	.318	.217
		Pearson Type II (m = .5)		
(.000,.000)	.057	.093 (.058)	.069	.080
(.141,.141)	.062	.109 (.087)	.095	.112
(.283,.283)	.094	.182 (.142)	.198	.225
(.424,.424)	.136	.334 (.266)	.390	.427
		Pearson Type VII (m = 2)		
(.000,.000)	.061	.066 (.044)	.069	.061
(.141,.141)	.082	.090 (.055)	.085	.091
(.283,.283)	.172	.148 (.116)	.178	.157
(.424,.424)	.335	.252 (.198)	.301	.316
		Pearson Type VII (m = .5, cauchy)		
(.000,.000)	.051	.029 (.019)	.064	.061
(.141,.141)	.096	.052 (.029)	.093	.103
(.283,.283)	.238	.101 (.076)	.185	.237
(.424,.424)	.431	.153 (.126)	.332	.432

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.
Nominal significance level: .05318.
Number of samples: 1000 from each distribution.
Sample sizes: m = 15, n = 15.

Table 7.11--continued.

Distributions:

Bivariate Pearson Type II

$$\frac{\Gamma(m+2)}{\Gamma(m+1)\pi} |\Sigma|^{-1/2}\{1-(\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})\}^{m} \text{ if } (\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta}) \leq 1,\ m > -1.$$

Bivariate Pearson Type VII

$$\frac{\Gamma(m)}{\Gamma(m-1)\pi} |\Sigma|^{-1/2}\{1+(\underset{\sim}{y}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\theta})\}^{-m},\ m > 1,\ \underset{\sim}{y} \in R^{2}.$$

Table 7.12
Two-Sample Monte Carlo Results for Bivariate Normal Mixtures
with m = 15, n = 15, and α = .05318

Bivariate median difference	Statistic				
	M	T_2^2		MB2	CS
	Mixture 1 ($p = .5$, $\mu_{11} = \mu_{12} = -1.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = 1.0$, $\sigma_{21} = \sigma_{22} = 1.0$, $\rho_2 = 0.0$)				
(.000,.000)	.053	.076	(.054)	.059	.065
(.014,.014)	.054	.074	(.057)	.059	.070
(.028,.028)	.066	.105	(.070)	.080	.089
(.042,.042)	.083	.151	(.107)	.112	.129
	Mixture 2 ($p = .5$, $\mu_{11} = \mu_{12} = -.5$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = .5$, $\sigma_{21} = \sigma_{22} = 1.0$, $\rho_2 = .9$)				
(.000,.000)	.045	.068	(.042)	.057	.048
(.141,.141)	.055	.069	(.052)	.062	.053
(.283,.283)	.071	.114	(.085)	.096	.081
(.424,.424)	.126	.194	(.147)	.159	.140
	Mixture 3 ($p = .5$, $\mu_{11} = \mu_{12} = -1.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = -.9$, $\mu_{21} = \mu_{22} = 1.0$, $\sigma_{21} = \sigma_{22} = 1.0$, $\rho_2 = 0.0$)				
(.000,.000)	.051	.071	(.052)	.057	.049
(.141,.141)	.058	.078	(.056)	.065	.049
(.283,.283)	.061	.111	(.077)	.097	.072
(.424,.424)	.086	.166	(.129)	.156	.114
	Mixture 4 ($p = .9$, $\mu_{11} = \mu_{12} = 0.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = 0.0$, $\sigma_{21} = \sigma_{22} = 20.0$, $\rho_2 = 0.0$)				
(.000,.000)	.051	.021	(.012)	.060	.042
(.141,.141)	.059	.021	(.011)	.072	.053
(.283,.283)	.091	.035	(.021)	.108	.103
(.424,.424)	.153	.060	(.033)	.166	.177

Table 7.12--continued.

Bivariate median difference	Statistic				
	M	T_2^2		MB2	CS
	Mixture 5 ($p = .9$, $\mu_{11} = \mu_{12} = 0.0$, $\sigma_{11} = \sigma_{12} = 1.0$, $\rho_1 = 0.0$, $\mu_{21} = \mu_{22} = 0.0$, $\sigma_{21} = 20.0$, $\sigma_{22} = 40.0$, $\rho_2 = 0.0$)				
(.000,.000)	.053	.021	(.012)	.060	.041
(.141,.141)	.063	.021	(.011)	.071	.054
(.283,.283)	.088	.031	(.020)	.096	.102
(.424,.424)	.130	.049	(.031)	.147	.179

Entries: Proportion of times that each test statistic exceeded its asymptotic critical value. Values in parentheses are rejection proportions for Hotelling's T_2^2 using the $F_{2,N-3}$ distribution.

Nominal significance level: .05318.

Number of samples: 1000 from each distribution.

Sample sizes: $m = 15$, $n = 15$.

Distributions: Bivariate normal mixture, choosing N_1 with probability p and N_2 with probability 1-p, where N_i, $i = 1, 2$, is bivariate normal with null mean (μ_{i1}, μ_{i2}) and covariance matrix

$$\begin{pmatrix} \sigma_{i1}^2 & \rho_i\sigma_{i1}\sigma_{i2} \\ \rho_i\sigma_{i1}\sigma_{i2} & \sigma_{i2}^2 \end{pmatrix}.$$

In conclusion, the two-sample affine-invariant procedures perform similarly to the corresponding one-sample procedures. Mardia's statistic performs best for heavy-tailed distributions and MB2 performs best for light-tailed distributions. Although Mardia's statistic does not perform well under bivariate normality, MB2 performs quite well relative to Hotelling's T_2^2. Mardia's statistic performs better than Hotelling's T_2^2 only for heavy-tailed distributions, but MB2 performs better than Hotelling's T_2^2 both for light- and heavy-tailed distributions. Also MB2 continues to perform well for mixed bivariate normal distributions in which the assumption of elliptical symmetry is violated. As is true for the one-sample statistic, MB, MB2 performs well for a wide variety of distributions. Hence, the use of MB2 in the bivariate two-sample location problem is well supported.

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Dawn Peters earned a bachelor's degree in psychology from the University of Florida in 1979. In 1981, she returned to the University of Florida to begin graduate studies, completing a master's degree in statistics in 1984. She plans to continue working in an academic setting upon completion of the Ph.D. requirements.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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